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Analysis of Growth, Inequality and Welfare Through the Interaction between Private Individual Labor-time and Public Spending on Education

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Abstract

This paper analyzes the effects of educational investments on long-term growth, inequality and welfare in an endogenous growth model in which a CES technology is specified for human capital accumulation. The objective is to explain how the degree of substitutability/complementarity between private and public educational investments affects economic variables. We show that the effect of public educational funding on growth is greater when both types of investments are complements than when they are substitutes. Moreover, in addition to the degree of substitutability/complementarity, the impact of public policy on inequality depends on the kinds of heterogeneity we introduce in the model. Finally, we find that the welfare-maximizing amount of public investment in education is increasing with the degree of substitutability/complementarity.

Keywords: Human Capital, Education, Growth, Welfare, Inequality

1. Introduction

Following the seminal work by Lucas (1988), many economists have developed both empirical and theoretical frameworks to show that education, as a prime component of human capital, plays a key role in explaining long-term economic growth. Looking at data from OECD (2013), we observe that, on average, OECD countries spend approximately 6.3 percent of their total resources in education. While a share around 83.6 percent of total spendings comes from public sources, the remaining share (16.4 percent) comes from private sources (see Table 1). Theoretically, some researchers emphasize how private investment affects the accumulation of human capital and then the productivity growth (see, e.g., Lucas, 1988), while other research focuses on the impacts of public educational spending on growth and income distribution (see, e.g., Zhang, 1996). In particular, there has been an increasing number of studies on both types of investments affecting economic variables (see, e.g., Glomm and Ravikumar, 1992; Blankenau and Simpson, 2004), but very few of them investigates the importance of the interaction between the two types of investment.

The goal of this paper is to analyze how the interaction between private and public investments affects economic variables, in particular, long-term economic growth, inequality and welfare. The reason is threefold. First is the issue of economic growth. A topic of interest is probably to analyze how the policy instruments used by the government affect economic growth via the process of human capital accumulation. In this paper, we will show that the outcome crucially depends on the degree of substitutability/complementarity between private and public investments in education.

Second, inequality is also an issue of deep concern. The recent growth literature still has no conclusion whether there is a trade-off between equality and growth. For example, Kaldor (1957) and Kuznets (1955) argue that, inequality, caused by a wider gap in savings between rich and poor, leads to a higher growth rate. In contrast, Tournemaine and Tsoukis (2009) and Aloi and Tournemaine (2013) show that there is not necessary a trade-off between equality and growth. In this paper, we show that, in addition to the importance of the degree of substitutability/ complementarity between private and public educational investments, the outcome regarding distribution depends on the way the modeler introduces heterogeneity between individuals (which in turn leads to inequality).

Table 1 Relative proportions of private and public expenditure on education for all levels of education in some OECD countries (2010)

Country	Share of private expenditures	Share of public expenditures	Total expenditures as a percentage of GDP
Australia	25.9	74.1	6.1
Austria	9.0	91.0	5.8
Belgium	5.2	94.8	6.6
Canada*	24.2	75.8	6.6
Czech Republic	12.3	87.7	4.7
Denmark	5.5	94.5	8.0
Finland	2.4	97.6	6.5
France	10.2	89.8	6.3
Iceland	9.6	90.4	7.8
Ireland	7.5	92.5	6.4
Italy	9.9	90.1	4.7
Japan	29.8	70.2	5.1
South Korea	38.4	61.6	7.6
Mexico	19.5	80.5	6.2
Netherlands	16.7	83.3	6.3
New Zealand	17.4	82.6	7.3
Poland	13.8	86.2	5.8
Portugal	7.4	92.6	5.8
Slovak Republic	15.8	84.2	4.6
Spain	14.6	85.4	5.6
Sweden	2.5	97.5	6.5
United Kingdom	31.4	68.6	6.5
United States	30.6	69.4	7.3
OECD average	**16.4**	**83.6**	**6.3**

* Year of reference 2009 instead of 2010.

Third, a major issue concerns welfare. The determination of the mix between private and public investments is essential. This paper analytically shows that the welfare-maximizing amount of public spending on education (i.e., the government size) is positively correlated with the degree of substitutability/complementarity between private and public investments.

Related to our work, some authors have developed models in which private and public investments in education are important factors for the determination of economic growth and welfare. For instance, using an overlapping generations (OLG) model, Blankenau and Simpson (2004) show that the long-run growth effect of public educational expenditure can either be positive or negative. Similarly to Barro (1990), public spending on education are funded through various nondistortionary and distortionary taxes (such as taxes on consumption, labor and capital income), leading to an inverted-U relationship between government spendings and growth. A notable assumption in their model, though, is that private and public investments are imperfect substitutes: Blankenau and Simpson (2004) use a Cobb-Douglas technology for human capital accumulation.

However, as pointed out by the report of OECD (2012), there is no evidence that private and public investments in education are substitutes: both can as well be imperfect complements. In this spirit and to take into account this feature, Arcalean and Schiopu (2010) formalize a Constant Elasticity of Substitution (CES) technology for the production of human capital. In other words, in their model, private and public spendings on education can either be substitutes or complements depending on the degree of substitutability/complementarity between the two types of investments. Arguing that private and public spendings on education can be complements, they seek to determine the amount of private and public resources which need to be invested in order to maximize growth.

In this paper, we go one step further. Indeed, we complement the analysis by Arcalean and Schiopu (2010) in three ways. First, we emphasize the impact of a change in the degree of substitutability/complementarity between private and public investments in human capital accumulation on economic growth. We show that the policy instrument used to collect the public fund allocated to education has an ambiguous effect on growth. The reason is that the policy instrument also impacts on private investment for which the outcome depends on the degree of substitutability/complementarity. In particular, we find that the growth effect of the policy instrument is greater when private and public investments are complements than when they are substitutes. The reason is that the combination of private and public investments leads to higher growth when they are complements than when they are substitutes.[1]

Second, we introduce and investigate various sources of heterogeneity between individuals to assess their impacts on economic variables. The noteworthy result from our study is that, beside the degree of substitutability/complementarity between private and public investments in human capital accumulation, the kind of heterogeneity introduced greatly matters with respect to a change in government policy. For instance, we show that the effect of the government policy on inequality depends on the degree of substitutability/complementarity between the two types of investments when people are differentiated by their learning abilities and by their motivation to work and study. However, such effect vanishes if individuals are differentiated by their skills for output production. In this case, we find that the policy has no impact on inequality.

Finally, we augment the analysis by Arcalean and Schiopu (2010) as we discuss welfare issues. We show that the welfare-maximizing size of the government is increasing with the degree of substitutability/complementarity between the two types of investments. This is because the reduction in the amount of their labor-time supply leads to an increase of their welfare via the greater amount of leisure time they enjoy.

The remainder of this paper is organized as follows. We introduce the model in section 2. In section 3, we derive and analyze the equilibrium properties in symmetry case with respect to growth and welfare. In this section, we also introduce and analyze the impact of heterogeneity. We conclude in section 4.

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[1] The result is consistent with the result of Blankenau and Simpson (2004) who argue that the impact of public investments on growth should be greater in a complementarity situation than in a substitutability one.

2. Model

Consider a closed economy in continuous time. Time, denoted by t, goes from zero to infinity. The economy is populated by a mass [0,1] of infinitely-lived individuals. For simplicity, we assume that there are two groups of identical individuals denoted by i, where i = 1, 2. Group 1 has a size of p and group 2 has a size of $1 - p$. We assume that the two groups are differentiated in terms of innate skills for the production of output, learning abilities for human capital accumulation, and preferences for leisure (see more detail below).

Each individual of group i has two activities. They work to produce an output, $Y_{i,t}$, through an output technology and attend academic activities to accumulate skills, $H_{i,t}$, through a human capital accumulation process (see, e.g., Lucas, 1988). Each individual is endowed with T units of time and $H_{i,0} > 0$ units of human capital. Production of individual human capital combines labor-time, $L^H_{i,t}$, and public investments through government expenditures, G_t. We assume that government expenditures are funded with a flat income tax rate and that the budget constraint of the government is balanced at each instant: $G_t = \tau \overline{Y}_t$, where $\overline{Y}_t = pY_{1,t} + (1 - p)\,Y_{2,t}$. Thus, each individual's remaining income, $(1 - \tau)\,Y_{i,t}$, is allocated to private consumption, $C_{i,t}$. The details of technologies and preferences are given below.

Following Tournemaine and Tsoukis (2009), we assume a linear technology for output production:

$$Y_{i,t} = A_i L^y_{i,t} H_{i,t}, \tag{1}$$

where $A_i > 0$ is an idiosyncratic productivity parameter and $L^y_{i,t}$ is the quantity of labor-time devoted to the production of output. Without loss of generality, we assume that individuals of group 1 have a higher level of innate skills than those of group 2: $A_1 \geq A_2$. The skills ratio, A_1/A_2, can then be interpreted as an indicator of skills heterogeneity: a higher (lower) value of A_1/A_2 means a higher (lower) level of skills heterogeneity. It will be important to keep this information in mind when we analyze the model under heterogeneity in section 3.3.

Following Kempf and Moizeau (2009), we assume that the technology of human capital accumulation is determined by private individual labor-time and public spending for education. However, we depart from the authors as the law of motion of human capital is expressed as a CES function given by:

$$\dot{H}_{i,t} = \phi_i[\beta(L^H_{i,t}H_{i,t})^\alpha + (1-\beta)(\tau\overline{Y}_t)^\alpha]^{1/\alpha}, \tag{2}$$

where $\phi_i > 0$ is an idiosyncratic parameter measuring the innate learning abilities, $0 < \beta < 1$ is the weight rate of private investment in education relative to public spending, $\alpha < 1$ is a measure of the degree of substitutability/complementarity between the two types of investments in the production of new units of human capital. As for the output technology, we assume that individuals have different learning abilities, ϕ_i, to accumulate human capital. Individuals of group 1 have a higher level of learning abilities than those of group 2: $\phi_1 \geq \phi_2$. The abilities ratio, ϕ_1 / ϕ_2, is an indicator of learning-abilities heterogeneity: a higher (lower) level of ϕ_1 / ϕ_2 means a higher (lower) level of learning-abilities heterogeneity.

Preferences of individuals are represented by the following utility function:

$$U_i = \int_0^\infty [\ln C_{i,t} + \delta_i(T - L^y_{i,t} - L^H_{i,t})]\, e^{-\rho t} dt, \tag{3}$$

where $\delta_i > 0$ is a measure of the marginal disutility of non-leisure time and $\rho > 0$ is the rate of time preference. In the same spirit as before, we assume that individuals of group 1 have a lower marginal disutility of non-leisure time than those of group 2: $\delta_1 \leq \delta_2$. In other words, we assume that individuals of group 1 have a higher motivation for work and education or feel less tired to allocate their time in these activities. As before, the motivation ratio, δ_1 / δ_2, is an indicator of individuals' motivation heterogeneity: a lower (higher) value of δ_1 / δ_2 means a higher (lower) level of heterogeneity in the motivation of individuals.

3. Equilibrium

In this section, we characterize the equilibrium. We proceed in three steps. First, we characterize the behavior of individuals. Second, we examine the steady state under symmetry and discuss its properties and policy implications on growth and welfare. Third, we analyze the steady state under heterogeneity and its impacts on growth, inequality and welfare.

3.1 Individuals' Problem

The problem of individuals is to choose consumption, $C_{i,t}$, labor-time devoted to work, $L^y_{i,t}$, and labor-time devoted to human capital accumulation, $L^H_{i,t}$, that maximize the lifetime utility function (3) subject to the output technology (1) and the resource constraint, given the law of motion of human capital (2) and the initial condition $H_{i,0} > 0$. After manipulations, the current value Hamiltonian of this problem is:

$$CVH_i = \ln C_{i,t} + \delta_i(T - L^y_{i,t} - L^H_{i,t}) + \lambda_{i,t}[(1-\tau)A_i L^y_{i,t} H_{i,t} - C_{i,t}] \\ + \mu_{i,t}\phi_i[\beta(L^H_{i,t}H_{i,t})^\alpha + (1-\beta)(\tau\overline{Y}_t)^\alpha]^{1/\alpha},$$

where $\lambda_{i,t}$ and $\mu_{i,t}$ are co-state variables associated with the resource constraint and the law of motion of human capital, respectively. The first order conditions are: $\partial CVH_i/\partial C_{i,t} = 0$, $\partial CVH_i/\partial L^y_{i,t} = 0$, $\partial CVH_i/\partial L^H_{i,t} = 0$, and $\partial CVH_i/\partial H_{i,t} = -\dot{\mu}_{i,t} + \rho\mu_{i,t}$. The transversality condition is $\lim_{t\to\infty}\mu_{i,t}H_{i,t}e^{-\rho t} = 0$. After simple computations, we obtain:

$$\frac{1}{C_{i,t}} = \lambda_{i,t}, \tag{4}$$

$$\frac{1}{L^y_{i,t}} = \delta_i, \tag{5}$$

$$\frac{\mu_{i,t}\beta(L^H_{i,t})^{\alpha-1}(H_{i,t})^\alpha \dot{H}_{i,t}}{[\beta(L^H_{i,t}H_{i,t})^\alpha + (1-\beta)(\tau\overline{Y}_t)^\alpha]} = \delta_i, \tag{6}$$

$$\rho = \frac{1}{\mu_{i,t}H_{i,t}} + \frac{\beta(L^H_{i,t}H_{i,t})^\alpha}{[\beta(L^H_{i,t}H_{i,t})^\alpha + (1-\beta)(\tau\overline{Y}_t)^\alpha]}\frac{\dot{H}_{i,t}}{H_{i,t}} + \frac{\dot{\mu}_{i,t}}{\mu_{i,t}}. \tag{7}$$

Expression (4) shows that the marginal utility of consumption equals the shadow price of output. Expression (5) states that the benefit of an additional unit of labor-time spent on the production of output equals its marginal cost measured by utility losses. Expression (6) shows that the marginal productivity of an additional amount of labor-time allocated to human capital accumulation is equal to its cost measured by utility losses. Finally, expression (7) is an asset-pricing equation indicating that the rate of time preference equals the rate of returns to education. This latter is given by its marginal productivity in the production of output and its return from future accumulation of human capital plus the change in the shadow price of education.

3.2 Steady state under symmetry

3.2.1 Characterization

Before investigating the role of heterogeneity, it is interesting to derive the basic properties of the model under symmetry, i.e. $A_1 = A_2$, $\phi_1 = \phi_2$, and $\delta_1 = \delta_2$.[2] This will allow us to obtain a first set of intuitions and help to understand the more complicated framework with differentiated individuals.

Under symmetry, the levels of output, consumption, human capital, labor-times are the same across individuals, so that we can ignore the subscript i. After manipulations of the first order conditions (4)-(7), we derive Proposition 1 where the symbol e denotes equilibrium values:

Proposition 1 *Under symmetry, there exists a unique steady-state equilibrium. The quantity of labor-time devoted to output production is given by:*

$$(L^y)^e = \frac{1}{\delta}. \tag{8}$$

The quantity of labor-time devoted to education is solution of:

$$\Delta[(L^H)^e] = \Psi[(L^H)^e], \tag{9}$$

where

$$\Delta[(L^H)^e] = \frac{\phi}{\delta}\{\beta[\delta(L^H)^e]^\alpha + (1-\beta)(\tau A)^\alpha\}^{\frac{1}{\alpha}}, \tag{10}$$

$$\Psi[(L^H)^e] = \rho\left\{\frac{\beta[\delta(L^H)^e]^{1+\alpha} + (1-\beta)(\tau A)^\alpha\delta(L^H)^e}{\beta[\delta(L^H)^e]^\alpha - (1-\beta)(\tau A)^\alpha\delta(L^H)^e}\right\}. \tag{11}$$

The common growth rate of human capital, consumption, and output, is given by:

$$g^e = \phi\{\beta[(L^H)^e]^\alpha + (1-\beta)[\tau A(L^y)^e]^\alpha\}^{1/\alpha}. \tag{12}$$

Proof. *See Appendix 5.2*

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[2] We show in Appendix 5.1 that there is no transitional dynamics in the case of symmetry.

Equation (8) shows that the steady-state equilibrium amount of labor-time devoted to output production, $(L^y)^e$, depends on the marginal disutility of non-leisure time, δ, while the steady-state equilibrium amount of labor-time allocated to human capital accumulation, $(L^H)^e$, is implicitly determined by equations (9)-(11). $\Delta(.)$ depicts a strictly increasing and concave function for any value of $\alpha < 1$. However, the shape of $\Psi(.)$ depends on the degree of substitutability/complementarity between private and public spendings on education, α. If the spendings are substitutes ($0 < \alpha < 1$), $\Psi(.)$ is strictly increasing and concave. In Appendix 5.2, we show that the slopes of $\Delta(.)$ and $\Psi(.)$ are different and intercept only once, guaranteeing the uniqueness of the steady-state solution.

If private and public spendings are complements ($\alpha < 0$) , $\Psi(.)$ is strictly increasing and convex. In this case, we show in Appendix 5.2 that $\Delta(.)$ and $\Psi(.)$ intercept only once. To clarify, we draw Figures 1.1-1.2 which depict the solution in the case of $0 < \alpha < 1$ and $\alpha < 0$, respectively. To proceed, we calibrate the model using benchmark parameter values to obtain an approximately credible growth rate of 2% (see, e.g., Arcalean and Schiopu, 2010; Tournemaine and Luangaram, 2012). For simplicity, we normalize $A = 1$ and the rests of the parameter values are provided in Table 2.

Table 2: Baseline parameter values

Description	Parameter	Value	Source/Remark
Marginal disutility of non-leisure time	δ	2.90	Tournemaine and Tsoukis (2014)
Rate of time preference	ρ	0.03	Barro and Sala-i-Martin (1992)
Productivity of education	ϕ	0.19	Manuelli and Seshadri (2010)
Income tax rate	τ	0.10	Tournemaine and Luangaram (2012)
Human capital time share	β	0.60	Erosa et al. (2010)
Degree of substitutability/ complementary between private and public investments	α	0.60	Arcalean and Schiopu (2010)

Figure 1.1: The Relationship between Growth Rate and Labor-time allocated to education ($\alpha = 0.6$)

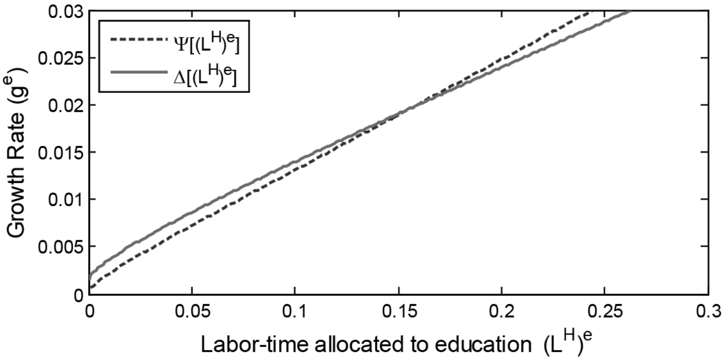
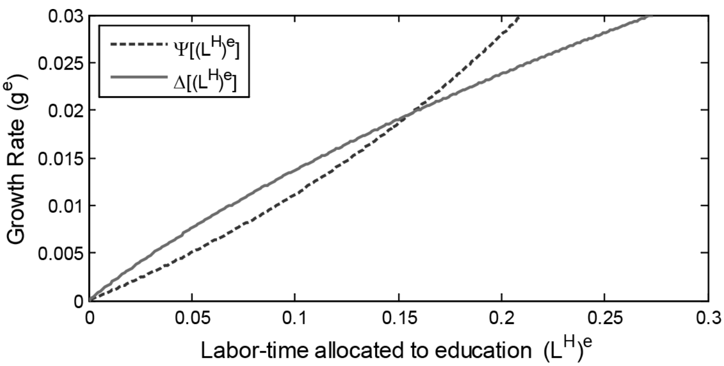


Figure 1.2: The Relationship between Growth Rate and Labor-time allocated to education ($\alpha = -0.6$)



3.2.2 Basic properties

In this section, we analyze the relationships between equilibrium variables and parameters of the model. We relegate the policy implications of the model to the next subsection. To proceed, we adapt Log-linearization with Taylor expansion by generating the two unknowns, $(L^H)^e$ and g^e, and exogenous parameters in matrices. This will allow us to find each pair of relationships between the unknowns and parameters (see Appendix 5.3). The results of the relationships are displayed in Table 3 below.

Table 3: Static Comparatives

	$x = \delta$	$x = \rho$	$x = \phi$	$x = A$
$\frac{d(L^H)^e}{dx}$	< 0	< 0	> 0	$>< 0$
$\frac{dg^e}{dx}$	< 0	< 0	> 0	$>< 0$

From Table 3, the marginal disutility of non-leisure time, δ, and the rate of time preference, ρ, have negative impacts on the equilibrium amount of labor-time allocated to human capital accumulation, $(L^H)^e$, and the growth rate, g^e. Intuitively, the idleness of individuals causes a lower amount of resources in human capital accumulation which reduces economic growth. Regarding ρ, the result is not surprising. As this parameter measures the preference of individuals for the present, a greater value of ρ means, ceteris paribus, that individuals prefer to increase their current consumption and leisure. On the contrary, the learning ability, ϕ, has positive impacts on $(L^H)^e$ and g^e. The reason is that it increases the productivity in the human capital sector.

The impact of the productivity parameter, A, on $(L^H)^e$ and g^e are more complex to analyze. We use a numerical method with a normalized CES production function of human capital (2)[3]. Figures 2.1-2.4 show that the relationships between the productivity parameter, A, and economic variables, $(L^H)^e$ and g^e, depend on the value of α. If $0 < \alpha < 1$ (i.e., private and public investments are substitutes), a higher level of innate skills, A, reduces resources in the productions of human capital and decreases economic growth. If $\alpha < 0$ (i.e., private and public investments are complements), the parameter A has positive impacts on $(L^H)^e$ and g^e. In other words, a higher level of innate skills

[3] De Jong (1967) and De Jong and Kumar (1972) pointed out that a CES production function does not satisfy the property of dimentional homogeneity. After, Klump and de La Grandville (2000) introduced a concept of normalisation (or re-parameterization) to eliminate arbitrary effects and inconsistent results. See our normalized CES production of human capital in Appendix 5.4.

increases resources in both the productions of output and human capital. In this case, the increased resources drive the growth up.

Consequently, the degree of substitutability/complementarity between private and public investments in education, α, plays a role in explaining how individual's innate skills, A, impacts on the amount of labor-time allocated to human capital accumulation, $(L^H)^e$, and the growth rate, g^e. However, the effects of his/her idleness, δ, preference for the present consumption, ρ, and learning abilities, ≚, on economic variables, $(L^H)^e$ and g^e, do not rely on α. In the next section, we will show how α plays a crucial role for policy implications on growth and welfare.

Figure 2.1: The relationship between $(L^H)^e$ and A where $\alpha = 0.6$

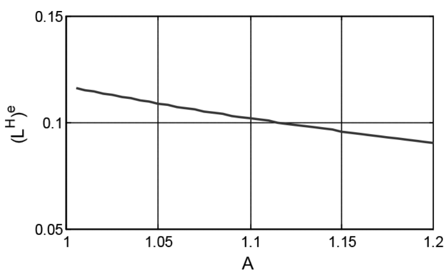


Figure 2.2: The relationship between $(L^H)^e$ and A where $\alpha = -0.6$

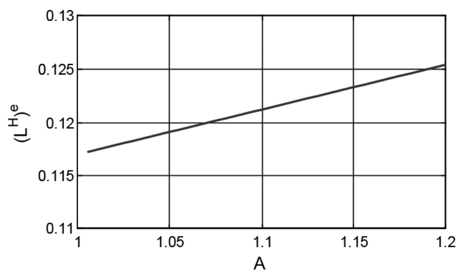


Figure 2.3: The relationship between g^e and A where $\alpha = 0.6$

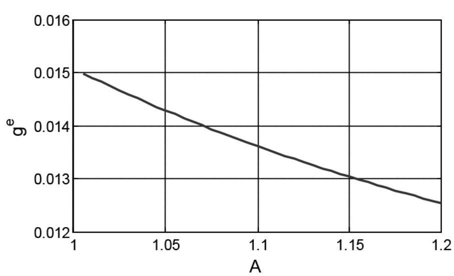
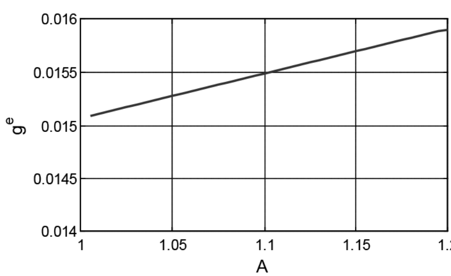


Figure 2.4: The relationship between g^e and A where $\alpha = -0.6$



3.2.3 Policy implications on growth and welfare

In this section, we discuss two issues: first, the impact of the policy instrument, τ, on growth. Second, we analyze its welfare implications.

3.2.3.1 Implications for growth

Using equation (12), the impact of τ on growth is given by:

$$\frac{dg^e}{d\tau} = \frac{g^e}{\beta[\delta(L^H)^e]^\alpha + (1-\beta)(\tau A)^\alpha}\left[(1-\beta)A^\alpha \tau^{\alpha-1} + \beta\delta^\alpha[(L^H)^e]^{\alpha-1}\frac{d(L^H)^e}{d\tau}\right]. \quad (13)$$

Expression (13) shows that public spending on education affects growth both directly (first term in bracket on the right hand side) and indirectly via its impact on the amount of labor-time, $(L^H)^e$, (second term in bracket on the right hand side). Obviously, the direct effect is always positive. However, the indirect effect can be of either sign. We have:

$$\frac{\tau}{(L^H)^e}\frac{d(L^H)^e}{d\tau} = \left(\frac{1}{1-\alpha}\right)\left(\frac{\rho}{g^e+\rho} - \alpha\right). \quad (14)$$

Expression (14) clearly shows that the impact of τ on $(L^H)^e$ depends on the value of α (i.e., on how α compares to $\rho/(g^e+\rho)$).[4] If $\alpha < 0$ (i.e., investments are complements), we have $d(L^H)^e/d\tau > 0$ which implies $dg^e/d\tau > 0$ (see Figures 3.2 and 3.4). For $\alpha < 0$, let $\hat{\alpha}$ be the value of the degree of substitutability/complementarity so that $d(L^H)^e/d\tau = 0$. If $0 < \alpha < \hat{\alpha}$ (i.e., investments are weak substitutes), we still have $d(L^H)^e/d\tau > 0$ implying $dg^e/d\tau > 0$. In this case, the combination of the two types of investments increases the amount of labor-time allocated to human capital accumulation, which leads to a higher growth rate. If $0 < \hat{\alpha} < \alpha < 1$ (i.e., investments are strong substitutes), we have $d(L^H)^e/d\tau < 0$ (see, e.g., Figure 3.1). However, the sign of $dg^e/d\tau$ can be either positive or negative, depending on the relative size of the direct effect and indirect effect discussed above. If the direct effect of the tax on growth through public spending dominates the indirect negative effect through private investment, we have $dg^e/d\tau > 0$. But, if the direct effect is dominated by the indirect one, we have $dg^e/d\tau < 0$ (see, e.g., Figure 3.3). This is consistent

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[4] To derive expression (14), we rewrite (34) in Appendix 5.3 by using (29) and (30).

with the result of Baier and Glomm (2001): at some point, a higher rate of tax on labor income leads to too little human capital being accumulated and a lower level of growth.

A noteworthy, somewhat interesting feature of the model is that the results are also consistent with empirical observations. Using data by de La Fuente and Doménech (2013) and OECD (2013), we can draw Figures 4.1 and 4.2. The former represents the relationship between average years of schooling and public expenditures on education. The latter represents the relationship between growth rate and public expenditures on education. We can see that these situations fit with the case where investments in human capital are complements ($\alpha < 0$) as shown in Figures 3.2 and 3.4. The question which needs to be answered now concerns the level of τ that maximizes welfare. We turn to this issue next.

Figure 3.1: The relationship between $(L^H)^e$ and τ where $\alpha = 0.6$

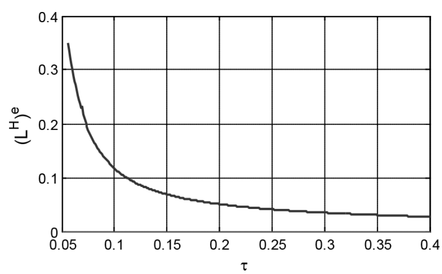


Figure 3.2: The relationship between $(L^H)^e$ and τ where $\alpha = -0.6$

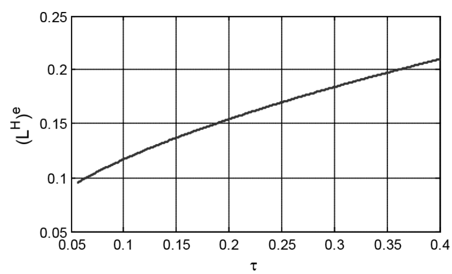


Figure 3.3: The relationship between g^e and τ where $\alpha = 0.6$

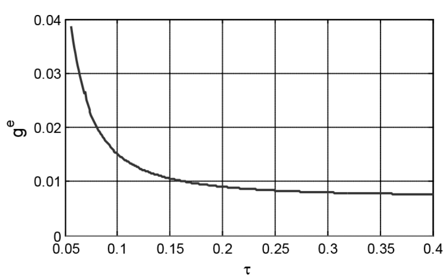


Figure 3.4: The relationship between g^e and τ where $\alpha = -0.6$

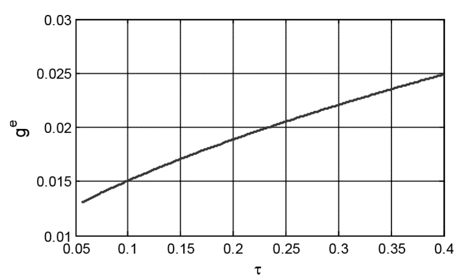


Figure 4.1: The relationship between private and public spendings on education in some OECD countries (2010)

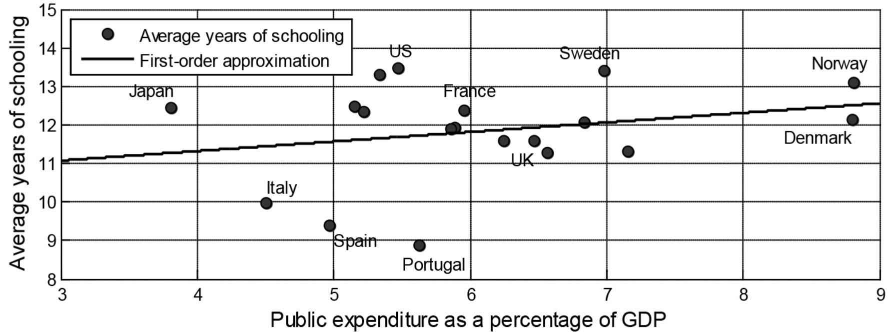
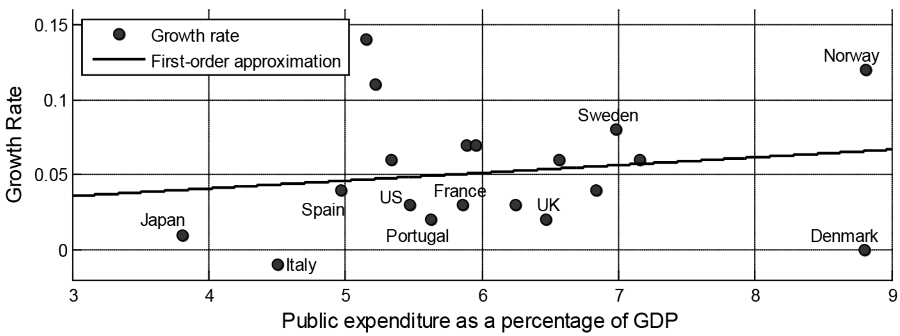


Figure 4.2: The relationship between growth and public spendings on education in some OECD countries (2010)



3.2.3.2 Implications for welfare

First, we determine the welfare impact of the policy instrument, τ. To proceed, we compute the lifetime utility of individuals. From equation (3), we obtain:

$$U = \frac{\ln[(1-\tau)(L^y)^e AH_0] + \delta[T - (L^y)^e - (L^H)^e]}{\rho} + \frac{g^e}{\rho^2}. \tag{15}$$

Differentiating equation (15) with respect to τ, we obtain:

$$\rho\frac{dU}{d\tau} = -\frac{1}{(1-\tau)} - \delta\frac{d(L^H)^e}{d\tau} + \frac{1}{\rho}\frac{dg^e}{d\tau}. \tag{16}$$

Expression (16) reveals that the tax rate, τ, has three effects on welfare (see the right hand side of the equation). The first term is unambiguously negative. It represents the consumption loss of a greater tax rate. The second and third terms can be of either sign as they represent the impact of τ on labor-time, $(L^H)^e$, and growth, g^e. Accordingly, we can establish the following Proposition.

Proposition 2 *For each value of* τ*, where* $0 < \tau < 1$*, there is a unique steady-state equilibrium: one of these maximizes welfare. Under symmetry, the welfare-maximizing tax rate denoted* τ^w *is solution of the following equation:*

$$\frac{1}{(1-\tau^w)} = -\delta\frac{d(L^H)^e}{d\tau} + \frac{1}{\rho}\frac{dg^e}{d\tau}.$$

Proof. *Readily follows from equation (16).*

The noteworthy feature of proposition 2 is that the welfare-maximizing tax rate, τ^w, indirectly depends on the degree of substitutability/complementarity between private and public investments in education, α, through $d(L^H)^e/d\tau$ and $dg^e/d\tau$.

Gathering Proposition 2 and the result of the previous subsection, we obtain Proposition 3.

Proposition 3 *Under the most plausible scenario of a condition* $\alpha < \hat{\alpha}$*, the welfare-maximizing tax rate,* τ^w*, is increasing with the degree of substitutability/complementarity between private and public investments in human capital accumulation,* α :

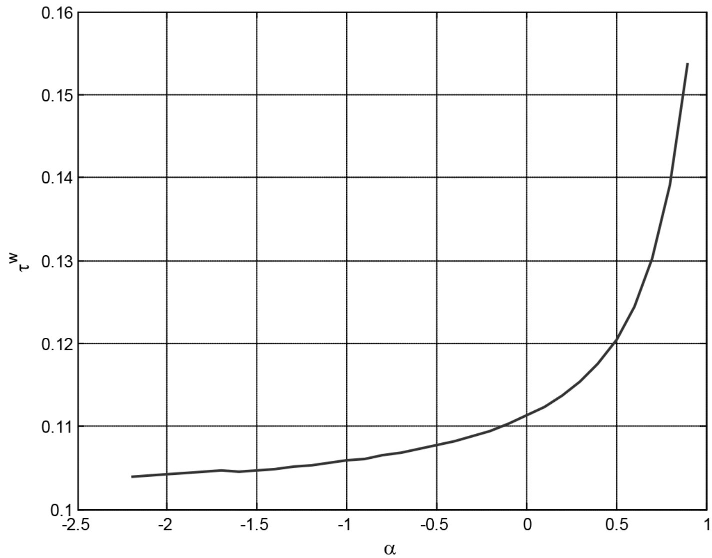
$$\frac{d\tau^w}{d\alpha} > 0.$$

Proof. *Readily follows from Proposition 2 and the property:* $d(L^H)^e/d\tau = 0$ *if* $\alpha = \hat{\alpha}$.

To illustrate the result of Proposition 3, we draw Figure 5 which represents the level of welfare- maximizing tax, τ^w, as a function of α. To proceed, we use the benchmark parameter values given in Table 2.

What Proposition 3 implicitly suggests, is that, the degree of substitutability/ complementarity, α, plays a key role with respect to the level of the policy instrument, τ, that the government should implement to maximize welfare. Formally, as private and public investments in education become more substitutable (i.e., α increases), the government should set a higher tax rate. This is because more substitutability leads to a reduction of labor-time devoted to human capital accumulation which increases individual's welfare through the effect on their consumption and leisure-time.

Figure 5: The relationship between welfare-maximizing tax (τ^w) and the degree of substitutability/complementarity (α)



3.3 Steady state under heterogeneity

In this section, we analyze the steady state under heterogeneity. As mentioned in section 2, we assume $A_1 \geq A_2$, $\phi_1 \geq \phi_2$ and $\delta_1 \leq \delta_2$. For simplicity, we restrict our attention to the steady state, so that growth rates, labor-times and individuals' variables ratio are constant. To proceed, we characterize the set of equations that we will use to analyze the impacts of heterogeneity and of the policy instrument on the equilibrium outcome. We will see that the degree of substitutability/ complementarity plays an important role regarding the kind of results we obtain.

3.3.1 Characterization

From the proof of Proposition 1 and the technology of production of output (1), the ratio of the amount of labor-time allocated to output production and the income ratio between individuals are respectively given by:

$$\tilde{L}^y = \frac{1}{\tilde{\delta}} \tag{17}$$

and

$$\tilde{Y} = \frac{\tilde{A}\tilde{H}}{\tilde{\delta}}, \tag{18}$$

where, for convenience, we denote by $\tilde{L}^y = L_1^y/L_2^y$; $\tilde{Y} = Y_{1,t}/Y_{2,t}$; $\tilde{A} = A_1/A_2$; $\tilde{\delta} = \delta_1/\delta_2$; and $\tilde{H} = H_{1,t}/H_{2,t}$. Using equation (2), we obtain:

$$g = \phi_1[\beta(L_1^H)^\alpha + (1-\beta)\tau^\alpha(p\frac{A_1}{\delta_1} + (1-p)\frac{A_2}{\delta_2\tilde{H}})^\alpha]^{1/\alpha} \tag{19}$$

and

$$g = \phi_2[\beta(L_2^H)^\alpha + (1-\beta)\tau^\alpha(p\frac{A_1\tilde{H}}{\delta_1} + (1-p)\frac{A_2}{\delta_2})^\alpha]^{1/\alpha}, \tag{20}$$

where $g = \dot{H}_{it}/H_{it}$. Moreover, the first-order conditions (4)-(7) imply:

$$\frac{g+\rho}{g} = (\frac{1+\delta_1 L_1^H}{\delta_1 L_1^H})\frac{\beta(L_1^H)^\alpha}{\beta(L_1^H)^\alpha + (1-\beta)\tau^\alpha(p\frac{A_1}{\delta_1} + (1-p)\frac{A_2}{\delta_2\tilde{H}})^\alpha} \tag{21}$$

and

$$\frac{g+\rho}{g} = (\frac{1+\delta_2 L_2^H}{\delta_2 L_2^H})\frac{\beta(L_2^H)^\alpha}{\beta(L_2^H)^\alpha + (1-\beta)\tau^\alpha(p\frac{A_1\tilde{H}}{\delta_1} + (1-p)\frac{A_2}{\delta_2})^\alpha} \tag{22}$$

Equations (17)-(22) constitute a system of six equations with six unknowns: $\tilde{L}^y, \tilde{Y}, L_1^H, L_2^H, \tilde{H}$ and g. The analysis of the impact of heterogeneity on these variables is presented next. Due to the complexity of the results, we use numerical methods with the benchmark parameter values in Table 2. We then study how a 10 percent reduction in heterogeneity (via a positive change in A_2 and ϕ_2 and a negative change in δ_2) affects $\tilde{L}^y$, $\tilde{Y}$, L_1^H, L_2^H, $\tilde{H}$

and g in the next section.[5] We proceed in a similar way to analyze the impact of a positive change in the policy instrument, τ, in section 3.3.3.

3.3.2 Impacts of heterogeneity

In this section, we show the impact of a reduction in heterogeneity on economic variables. A brief summary of results is given in Table 4 and discussed in the following subsections.

Table 4: Effects of 10 percent changes in parameters on economic variables

10% Change of parameter	Situation	Gap of L_i^H	Gap of H_i	Income Inequality	Growth
$\uparrow A_2$	$0 < \alpha < 1$	0	0	−9.09%	+1.73%
	$\alpha < 0$	0	0	−9.09%	+4.42%
$\uparrow \phi_2$	$0 < \alpha < 1$	−26.25%	−66.07%	−66.07%	+16.68%
	$\alpha < 0$	−3.91%	−15.78%	−15.78%	+7.96%
$\uparrow \delta_2$	$0 < \alpha < 1$	−49.08%	−167.57%	−194.33%	+14.29%
	$\alpha < 0$	−5.86%	−7.92%	−18.71%	+7.36%

Notes: (–) denotes reducing gaps of variables and (+) denotes expanding gaps.

3.3.2.1 Effects of skills heterogeneity: $\tilde{A}$

In this subsection, we let $\tilde{L}^H = L_1^H / L_2^H$, $\delta_1 = \delta_2$, $\phi_1 = \phi_2$, and assume that A_2 increases from 0.9091 to 1, which corresponds to a reduction of $\tilde{A}$. Results are shown in Figures 6.1-6.6 and 7.1-7.6. Intuitions are as follows.

As A_2 increases, individuals of group 2 become more productive in the output sector. As a result, they produce a greater amount of output, synonymous of a greater amount of tax collected that can be invested in the human capital accumulation process of both types of individuals. In this case, individuals choose to increase the amount of labor-time they allocate to human capital accumulation (see Figures 6.1-6.4).

5 Throughout, we omit the analysis of $\tilde{L}^y$. This variable effectively depends only on the motivation ratio, $\tilde{\delta}$. In other words, the higher level of the motivation heterogeneity, the larger quantity difference between individual labor supplies.

Interestingly, the ratio of labor-time allocated to human capital accumulation between individuals, $\tilde{L}^H$, is constant approximately at one when $0 < \alpha < 1$ and $\alpha < 0$ (see Figures 6.5-6.6). This implies that there is no gap between L_1^H and L_2^H for any value of $\alpha < 1$.[6] The reason is that a change in skills heterogeneity has equivalent effects for both types of individuals.

Figure 6.1: The relationship between L_1^H and A_2 where α = 0.6

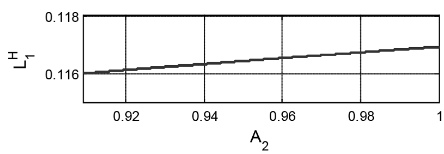


Figure 6.2: The relationship between L_1^H and A_2 where α = –0.6

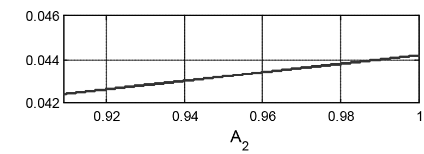


Figure 6.3: The relationship between L_2^H and A_2 where α = 0.6

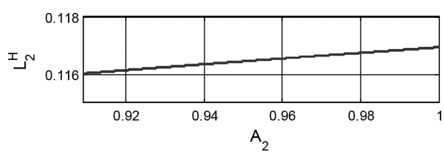


Figure 6.4: The relationship between L_2^H and A_2 where α = –0.6

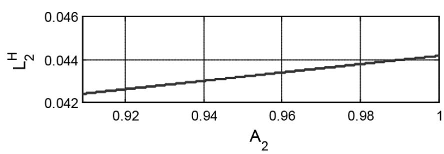


Figure 6.5: The relationship between private-investment ratio and A_2 where α = 0.6

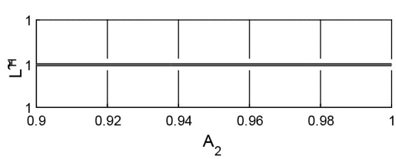
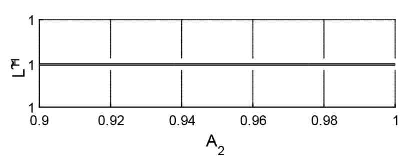


Figure 6.6: The relationship between private-investment ratio and A_2 where α = –0.6



[6] Note that if the ratio $\tilde{L}_H$, $\tilde{H}$ or $\tilde{Y}$ is equal to 1, there is no gap between its variable values (i.e., $L_1^H = L_2^H$, $H_1 = H_2$, or $Y_1 = Y_2$). But if it is different from 1, there exists a gap. For example, $\tilde{L}_H > 1$ means that individuals of group 1 spend more time allocated to education than those of group 2, while $\tilde{L}_H < 1$ means the reversed. Thus, if $\tilde{L}_H$ is increasing (decreasing) from the value below (above) one and is approaching to one, the relative gap between L_1^H and L_2^H shrinks and closes until $L_1^H = L_2^H$.

These results carry on for the relative amount of human capital between individuals, $\tilde{H}$, (see Figures 7.1-7.2). That is, there is no inequality gap in human capital between individuals for any value of $\alpha < 1$. Moreover, the effects on growth are also straightforward: it increases because of the higher level of private investments by both types of individuals (see Figures 7.3-7.4). However, the percentage increase in growth is positively greater in the complementarity case compared to that in the substitutability case (see Table 4).

As shown in equations (18)-(22), the effect of A_2 on income inequality, $\tilde{Y}$, is directly determined by A_2 itself and indirectly through $\tilde{H}$. Obviously, the direct effect is negative. However, there is no indirect effect (see Figures 7.1-7.2). Thus, Figures 7.5-7.6 show that a reduction of skills heterogeneity causes a narrowing of the inequality gap between individuals whether private and public investments are either substitutes or complements.

Figure 7.1: The relationship between human capital heterogeneity and A_2 where $\alpha = 0.6$

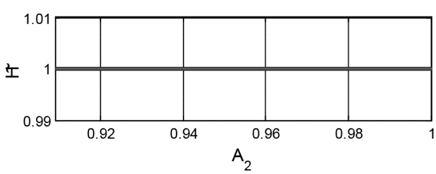


Figure 7.2: The relationship between human capital heterogeneity and A_2 where $\alpha = -0.6$

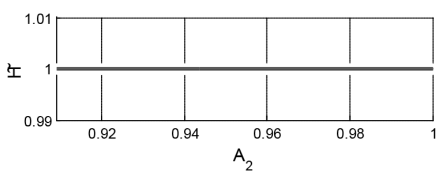


Figure 7.3: The relationship between growth and A_2 where $\alpha = 0.6$

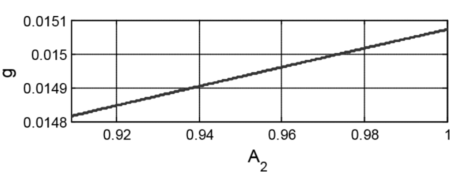


Figure 7.4: The relationship between growth and A_2 where $\alpha = 0.6$

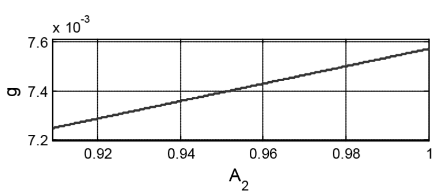


Figure 7.5: The relationship between inequality and A_2 where $\alpha = 0.6$

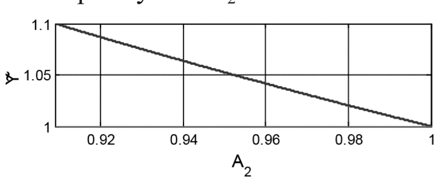
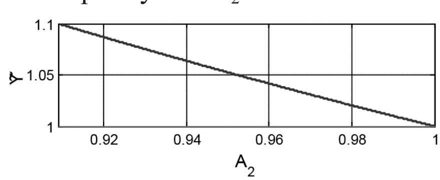


Figure 7.6: The relationship between inequality and A_2 where $\alpha = 0.6$



3.3.2.2 Effects of learning-abilities heterogeneity: $\tilde{\phi}$

We let $A_2 = A_1$, $\delta_2 = \delta_1$ and assume that ϕ_2 increases from 0.1727 to 0.19, which corresponds to a reduction of $\tilde{\phi}$. Results are shown in Figures 8.1-8.6 and 9.1-9.6. As before, intuitions are as follows.

In this case, individuals of group 2 become more productive in accumulating human capital as ϕ_2 increases (see Figures 8.3-8.4). The additional amount of human capital they produce allows them to also increase the production of output. Then, similarly as before, this means that the government can collect more funds to invest in the human capital process of both types of individuals. This is the reason why L_1^H increases as well (see Figures 8.1-8.2).

However, we note that the effect is more pronounced for individuals of group 2 (group 1) when private and public investments in human capital accumulation are substitutes (complements) (see Figures 8.5-8.6). The noteworthy feature is that the inequality gap between L_1^H and L_2^H shrinks for any value of $\alpha < 1$. It therefore leads to a reduction of human-capital inequality, $\tilde{H}$, (see Figures 9.1-9.2). Moreover, the effect on growth is unambiguously positive due to the increases in private and public investments in human capital accumulation (see Figures 9.3-9.4). From equation (18), we finally conclude that the effects of a reduction of $\tilde{\phi}$ (through an increase of ϕ_2) on the income gap between individuals, $\tilde{Y}$, are the same as those described for $\tilde{H}$ (see Figures 9.5-9.6).

Figure 8.1: The relationship between L_1^H and ϕ_2 where $\alpha = 0.6$

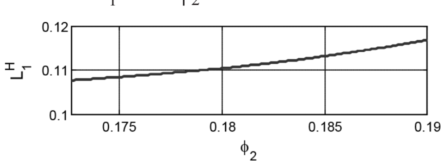


Figure 8.2: The relationship between L_1^H and ϕ_2 where $\alpha = -0.6$

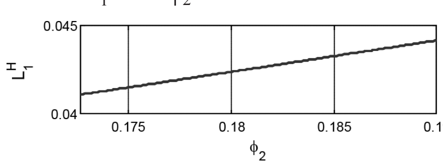


Figure 8.3: The relationship between L_2^H and ϕ_2 where $\alpha = 0.6$

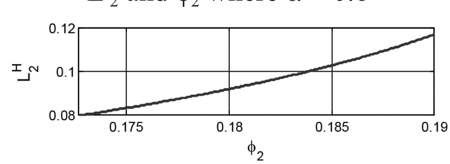


Figure 8.4: The relationship between L_2^H and ϕ_2 where $\alpha = -0.6$

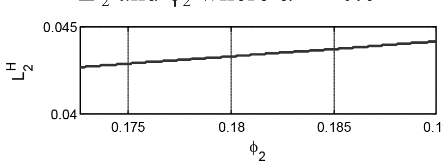


Figure 8.5: The relationship between private-investment ratio and ϕ_2 where $\alpha = 0.6$

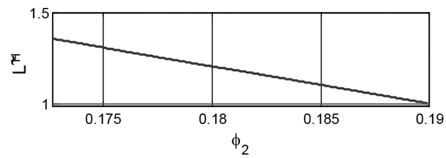


Figure 8.6: The relationship between private-investment ratio and ϕ_2 where $\alpha = -0.6$

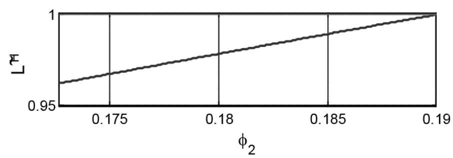


Figure 9.1: The relationship between human capital heterogeneity and ϕ_2 where $\alpha = 0.6$

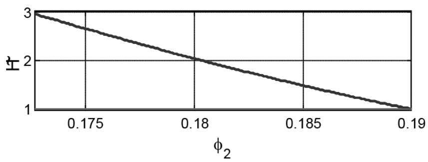


Figure 9.2: The relationship between human capital heterogeneity and ϕ_2 where $\alpha = -0.6$

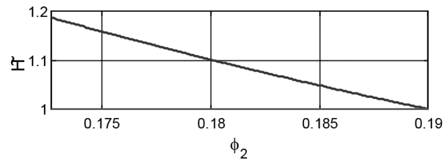


Figure 9.3: The relationship between growth and ϕ_2 where $\alpha = 0.6$

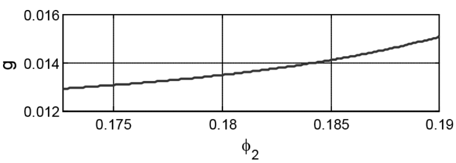


Figure 9.4: The relationship between growth and ϕ_2 where $\alpha = -0.6$

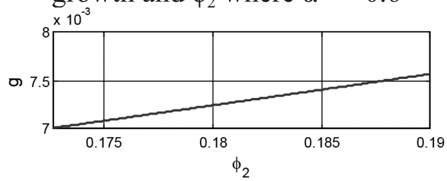


Figure 9.5: The relationship between inequality and ϕ_2 where $\alpha = 0.6$

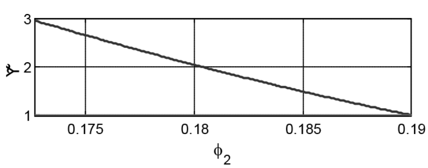
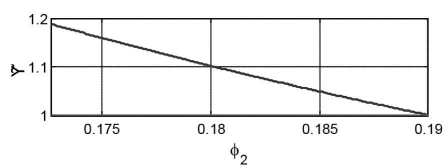


Figure 9.6: The relationship between inequality and ϕ_2 where $\alpha = -0.6$



According to Table 4, the result reveals that there are larger (lower) gaps of labor-time, human capital and income between individuals in the substitutability (complementarity) case when individuals are differentiated in learning abilities. Thus, a closing gap of learning-abilities heterogeneity leads to greater (smaller) percentage changes in inequality gaps and growth rate when private and public investments are substitutes (complements).

3.3.2.3 Effects of motivation heterogeneity: $\tilde{\delta}$

Now, we let $A_2 = A_1$, $\phi_2 = \phi_1$ and assume that δ_2 decreases from 3.19 to 2.90, which corresponds to a higher level of $\tilde{\delta}$, i.e., a reduction in motivation heterogeneity between individuals. As δ_2 decreases, individuals of group 2 are more motivated to work in the output sector and to allocate more labor-time to human capital accumulation. This leads to an increase of L_2^H (see Figures 10.3-10.4). As explained previously, this implies a higher amount of revenue collected tax which can be allocated to the human capital accumulation process of both types of individuals. As a result, L_1^H increases as well (see Figure 10.1-10.2).

Figure 10.1: The relationship between L_1^H and δ_2 where $\alpha = 0.6$

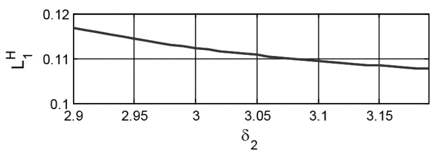


Figure 10.2: The relationship between L_1^H and δ_2 where $\alpha = -0.6$

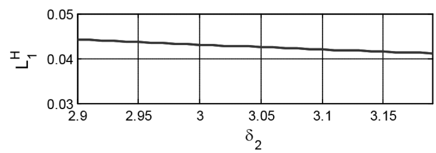


Figure 10.3: The relationship between L_2^H and δ_2 where $\alpha = 0.6$

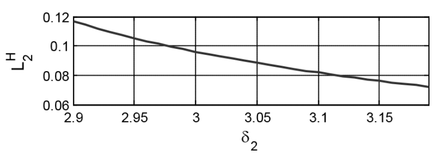


Figure 10.4: The relationship between L_2^H and δ_2 where $\alpha = -0.6$

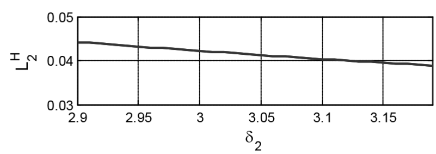


Figure 10.5: The relationship between private-investment ratio and δ_2 where $\alpha = 0.6$

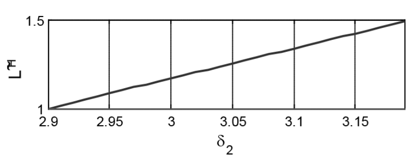
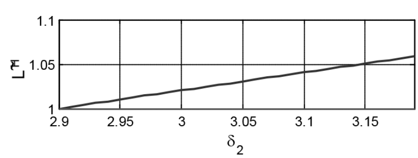


Figure 10.6: The relationship between private-investment ratio and δ_2 where $\alpha = -0.6$



In this case, the effect is more pronounced for individuals of group 2 whether private and public investments in human capital accumulation are either substitutes or complements (see Figures 10.5-10.6). Thus the inequality gap between L_1^H and L_2^H shrinks as δ_2 decreases for any value of $\alpha < 1$. As a result, the inequality gap in human capital between individuals, $\tilde{H}$, reduces (see Figures 11.1-11.2). Moreover, the effect on growth is always positive due to the higher levels of private and public investments in the human capital accumulation process of both types of individuals (see Figures 11.3-11.4).

Equations (18)-(22) show that there are direct and indirect effects of δ_2 on income inequality between individuals, $\tilde{Y}$. A lower level of δ_2 decreases $\tilde{Y}$ indirectly through a fall in $\tilde{H}$ as reported in Figures 11.5-11.6. Quantitatively, Table 4 shows that the percentage changes of all inequality gaps are greater (smaller) when private and public investments are substitutes (complements). The intuition is similar to the previous case: motivation heterogeneity causes larger gaps of inequality when the two types of investments are substitutes than they are complements. In the case of substitutability, moreover, a 10 percentage reduction in motivation heterogeneity causes greater changes in inequality compared to those caused by a 10 percentage reduction of learning-abilities heterogeneity.

Figure 11.1: The relationship between human capital heterogeneity and δ_2 where $\alpha = 0.6$

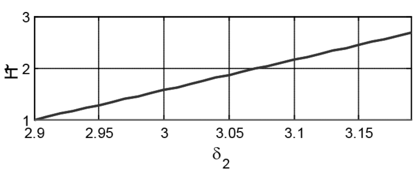


Figure 11.2: The relationship between human capital heterogeneity and δ_2 where $\alpha = -0.6$

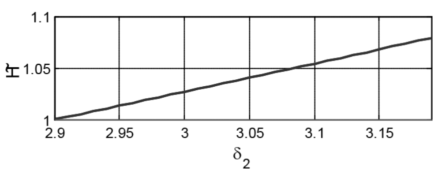


Figure 11.3: The relationship between growth and δ_2 where $\alpha = 0.6$

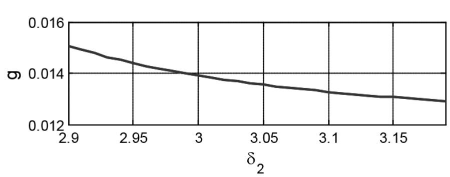


Figure 11.4: The relationship between growth and δ_2 where $\alpha = -0.6$

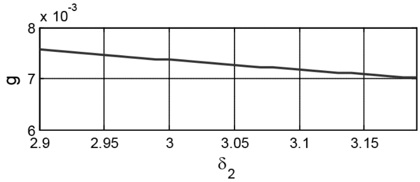


Figure 11.5: The relationship between inequality and δ_2 where $\alpha = 0.6$

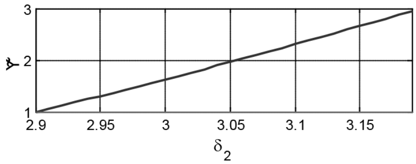
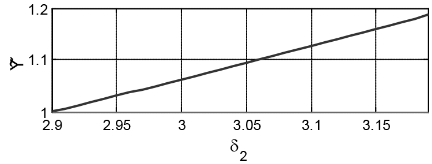


Figure 11.6: The relationship between inequality and δ_2 and δ_2 where $\alpha = -0.6$



Gathering the results depicted in this section, we find that the reduction in all heterogeneity yields qualitative equivalent results except $\tilde{L}^H$ in the case of the reduction in learning-abilities heterogeneity in Figures 8.5-8.6. Additionally, Table 4 shows that quantitative results depend on the degree of substitutability/complementarity between private and public investments in human capital accumulation. Specifically, the motivation heterogeneity exhibits the largest gaps in labor-time, human capital and income when private and public investments are substitutes, leading to the greatest percentage changes in these variables compared to the two other cases of heterogeneity. In addition, the reductions of heterogeneity in learning abilities and motivation give higher percentage increases in growth rate when the two types of investments are substitutes than when they are complements, while that of skills heterogeneity gives the reversed result.

3.3.3 Impact of policy instrument

In this section, we analyze the policy implications on inequality. We distinguish three cases: (i) skills heterogeneity ($A_1 > A_2$, $\phi_1 = \phi_2$ and $\delta_1 = \delta_2$) (ii) learning-abilities heterogeneity ($A_1 = A_2$, $\phi_1 > \phi_2$ and $\delta_1 = \delta_2$) and (iii) motivation heterogeneity ($A_1 = A_2$, $\phi_1 = \phi_2$ and $\delta_1 < \delta_2$). Moreover, we assume that τ increases from 0.1 to 0.4 and there is a 10 percentage gap between individuals of group 1 and 2 in each case of heterogeneity. A brief summary of the results is shown in Table 5.

Table 5: Effects of policy instrument on economic variables under heterogeneity

Case	Situation	Gap of L_i^H	Gap of H_i	Income Inequality	Growth
Skills heterogeneity $A_1 = 1, A_2 = 0.9091$, $\phi_1 = \phi_2 = 0.19$, $\delta_1 = \delta_2 = 2.9$	$0 < \alpha < 1$	0	0	0	+36.27%
	$\alpha < 0$	0	0	0	+240.98%
Learning-abilities heterogeneity $A_1 = 1, A_2 = 1$, $\phi_1 = 0.19, \phi_2 = 0.1727$, $\delta_1 = \delta_2 = 2.9$	$0 < \alpha < 1$	−10.59%	−64.63%	−64.63%	+29.86%
	$\alpha < 0$	+0.47%	+0.59%	+0.59%	+241.45%
Motivation heterogeneity $A_1 = 1, A_2 = 1$, $\phi_1 = \phi_2 = 0.19$, $\delta_1 = 2.90, \delta_2 = 3.19$	$0 < \alpha < 1$	−10.59%	−64.62%	+64.62%	+29.86%
	$\alpha < 0$	−0.47%	+0.59%	−0.59%	+241.44%

Notes: (−) denotes reducing gaps of variables and (+) denotes expanding gaps.

3.3.3.1 The case of skills heterogeneity

Similarly to the symmetry case, a higher level of τ causes a decrease (increase) of labor-time allocated to human capital accumulation for both types of individuals if $0 < \alpha < 1$ ($\alpha < 0$) (see Figures 12.1-12.4). However, as τ increases, there is no gap between L_1^H and L_2^H for any value of $\alpha < 1$ (see Figures 12.5-12.6). This is because the effects on labor-time of the two types of individuals are equivalent. This leads to no inequality gap in human capital, $\tilde{H}$, and a constant gap of income, $\tilde{Y}$, between individuals (see Figures 13.1-13.2 and 13.5-13.6).

However, as τ rises, the growth rate increases for any value of $\alpha < 1$ (see Figures 13.3-13.4). The reason of the increase in growth rate when $\alpha < 0$ is obviously due to the higher level of the increases in private and public investments in human capital accumulation. Specifically, the rise in growth rate when $0 < \alpha < 1$ implies that the direct effect of tax rate dominates the indirect one through the reduction of private investments of both types of individuals (see section 3.2.3.1).

Quantitatively, a rise in tax rate causes almost seven fold greater percentage increase in growth rate when private and public investments are complements than when they are substitutes (see Table 5). The reason is that the combination of the increases in private investments and public spending stimulates economic growth faster in the complementarity case.

Figure 12.1: The relationship between L_1^H and τ where $A_1 > A_2$ and α = 0.6

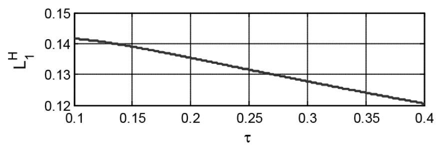


Figure 12.2: The relationship between L_1^H and τ where $A_1 > A_2$ and α = –0.6

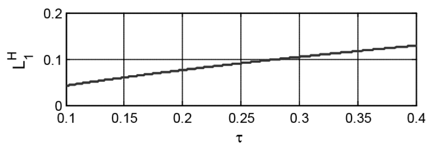


Figure 12.3: The relationship between L_2^H and τ where $A_1 > A_2$ and α = 0.6

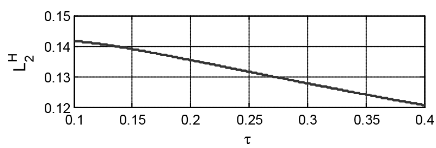


Figure 12.4: The relationship between L_2^H and τ where $A_1 > A_2$ and α = –0.6

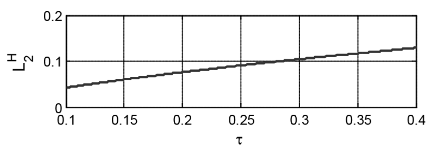


Figure 12.5: The relationship between private-investment ratio and τ where $A_1 > A_2$ and α = 0.6

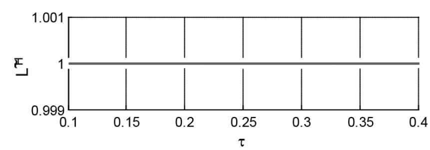


Figure 12.6: The relationship between private-investment ratio and τ where $A_1 > A_2$ and α = –0.6

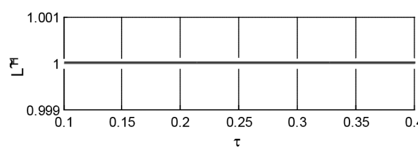


Figure 13.1: The relationship between human capital heterogeneity and τ where $A_1 > A_2$ and α = 0.6

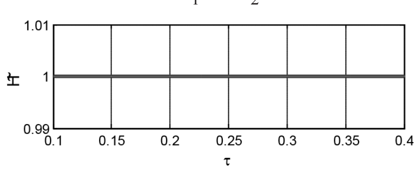


Figure 13.2: The relationship between human capital heterogeneity and τ where $A_1 > A_2$ and α = –0.6

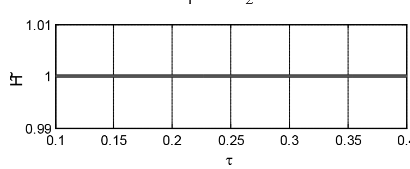


Figure 13.3: The relationship between growth and τ where $A_1 > A_2$ and α = 0.6

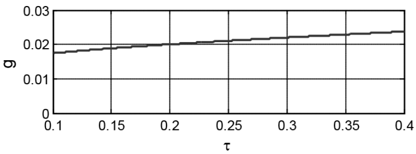


Figure 13.4: The relationship between growth and τ where $A_1 > A_2$ and α = –0.6

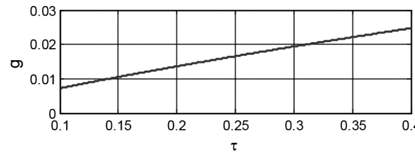


Figure 13.5: The relationship between inequality and τ where $A_1 > A_2$ and α = 0.6

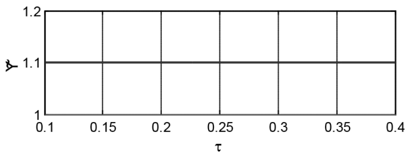
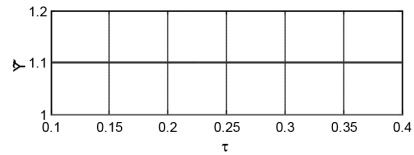


Figure 13.6: The relationship between inequality and τ where $A_1 > A_2$ and α = –0.6



3.3.3.2 The case of learning-abilities heterogeneity

A higher level of τ causes a decrease (increase) of labor-time allocated to human capital accumulation of both types of individuals if $0 < \alpha < 1$ ($\alpha < 0$). In this case, the effect on labor-time is more pronounced for individuals of group 1 (group 2) if $0 < \alpha < 1$ ($\alpha < 0$) (see Figures 14.1-14.6). Thus, $\tilde{L}_H$ is decreasing for any value of $\alpha < 1$. However, the relative gap between L_1^H and L_2^H shrinks (expands) when $0 < \alpha < 1$ ($\alpha < 0$).

Figure 14.1: The relationship between L_1^H and τ where $\phi_1 > \phi_2$ and α = 0.6

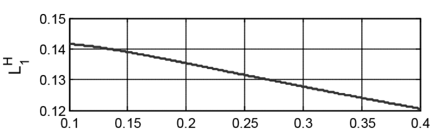


Figure 14.2: The relationship between L_1^H and τ where $\phi_1 > \phi_2$ and α = –0.6

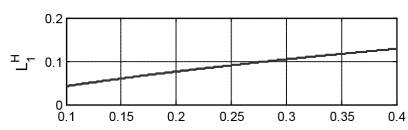


Figure 14.3: The relationship between L_2^H and τ where $\phi_1 > \phi_2$ and $\alpha = 0.6$

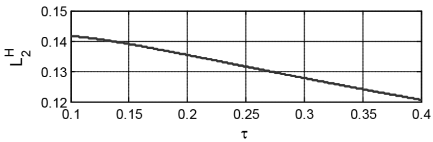


Figure 14.4: The relationship between L_2^H and τ where $\phi_1 > \phi_2$ and $\alpha = -0.6$

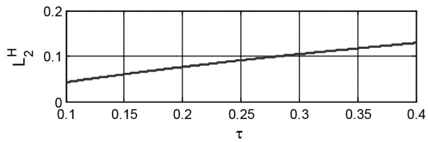


Figure 14.5: The relationship between private-investment ratio and τ where $\phi_1 > \phi_2$ and $\alpha = 0.6$

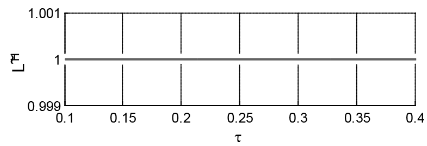


Figure 14.6: The relationship between private-investment ratio and τ where $\phi_1 > \phi_2$ and $\alpha = -0.6$

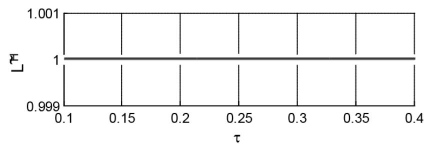


Figure 15.1: The relationship between human capital heterogeneity and τ where $\phi_1 > \phi_2$ and $\alpha = 0.6$

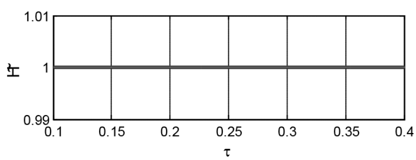


Figure 15.2: The relationship between human capital heterogeneity and τ where $\phi_1 > \phi_2$ and $\alpha = -0.6$

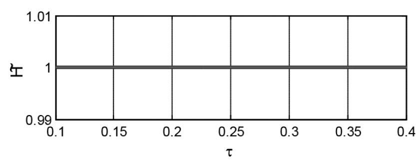


Figure 15.3: The relationship between growth and τ where $\phi_1 > \phi_2$ and $\alpha = 0.6$

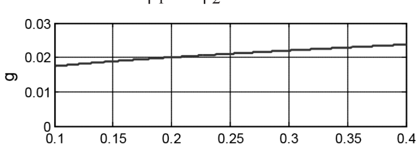


Figure 15.4: The relationship between growth and τ where $\phi_1 > \phi_2$ and $\alpha = -0.6$

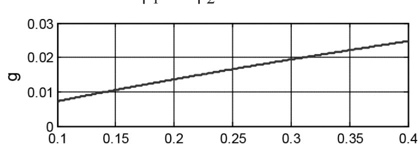


Figure 15.5: The relationship between inequality and τ where $\phi_1 > \phi_2$ and $\alpha = 0.6$

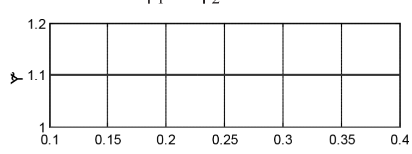
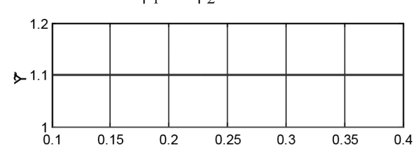


Figure 15.6: The relationship between inequality and τ where $\phi_1 > \phi_2$ and $\alpha = -0.6$



These results extend to the inequality gaps in human capital, $\tilde{H}$ and in income, $\tilde{Y}$, (see Figures 15.1-15.2 and 15.5-15.6). Regardless of the sign, the substitutability case exhibits greater percentage changes in all inequality gaps than the complementarity case (see Table 5). Again, the reason is that there is a larger (lower) gap in inequality in the substitutability (complementarity) case when individuals are differentiated in learning abilities. Thus, policy instrument causes larger effects on the gaps when private and public investments are substitutes than when they are complements.

Similarly to the intuition in section *3.3.3.1*, the growth rate is increasing with τ for any value of $\alpha < 1$. Moreover, a rise in tax rate causes approximately eight fold greater percentage increase in growth rate when private and public investments are complements than when they are substitutes (see Table 5).

3.3.3.3 The case of motivation heterogeneity

The effects of the policy instrument on labor-time allocated to human capital accumulation process of both types of individuals is similar to the two previous cases (see Figures 16.1-16.6). However, here, $\tilde{L}_H$ is decreasing for any value of $\alpha < 1$. This is because the effect is more pronounced for individuals of group 1 (group 2) when $0 < \alpha < 1$ ($\alpha < 0$). Comparing to the case of learning-abilities heterogeneity, the gap narrows rather than expands wh en $\alpha < 0$. This shows a great difference because an increase in tax rate causes a higher level of L_1^H than that of L_2^H when individuals of group 1 have a higher motivation for work and education, while it causes a lower level of L_1^H than that of L_2^H when they have a higher level of learning abilities.

Figure 16.1: The relationship between L_1^H and τ where $\delta_1 > \delta_2$ and $\alpha = 0.6$

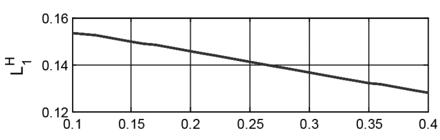


Figure 16.2: The relationship between L_1^H and τ where $\delta_1 > \delta_2$ and $\alpha = -0.6$

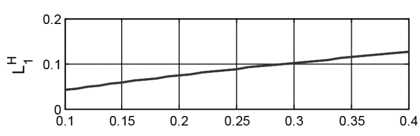


Figure 16.3: The relationship between L_2^H and τ where $\delta_1 > \delta_2$ and $\alpha = 0.6$

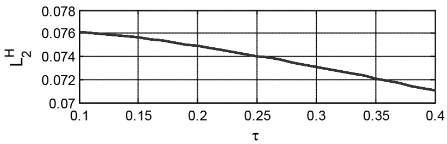


Figure 16.4: The relationship between L_2^H and τ where $\delta_1 > \delta_2$ and $\alpha = -0.6$

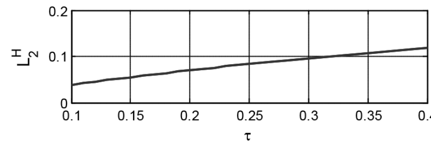


Figure 16.5: The relationship between private-investment ratio and τ where $\delta_1 > \delta_2$ and $\alpha = 0.6$

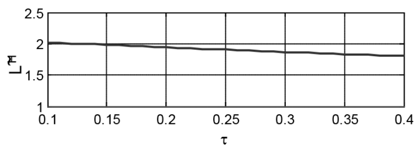
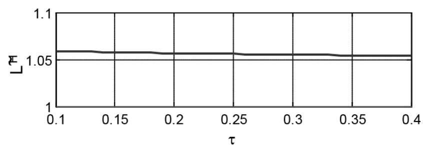


Figure 16.6: The relationship between private-investment ratio and τ where $\delta_1 > \delta_2$ and $\alpha = -0.6$



The effect on $\tilde{L}_H$ then extends to the inequality gaps in human capital, $\tilde{H}$, and in income, $\tilde{Y}$, only when $0 < \alpha < 1$ (see Figures 17.1 and 17.5). Interestingly, Figures 17.2 and 17.6 show that $\tilde{H}$ and $\tilde{Y}$ do not reduce but rise when $\alpha < 0$. The reason is that the effect of τ through the motivation heterogeneity dominates the effect of τ through the reduction of $\tilde{L}_H$. Quantitatively, the substitutability case exhibits, regardless of the sign, greater percentage changes in all inequality gaps than the complementarity case (see Table 5). The intuition is similar to the previous case: motivation heterogeneity causes larger gaps of inequality between individuals when the private and public investments are substitutes than they are complements.

Finally, Figures 17.3-17.4 illustrate that an increase of τ causes a higher level of growth, g, when $0 < \alpha < 1$ and $\alpha < 0$. Note that the quantitative result on growth is similar to the previous case: a rise in tax rate causes approximately eight fold greater percentage increase in growth rate when private and public investments in human capital accumulation are complements than when they are substitutes (see Table 5). The intuitions are also similar to the two previous cases.

Figure 17.1: The relationship between human capital heterogeneity and τ where $\delta_1 > \delta_2$ and $\alpha = 0.6$

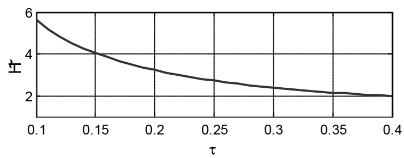


Figure 17.2: The relationship between human capital heterogeneity and τ where $\delta_1 > \delta_2$ and $\alpha = -0.6$

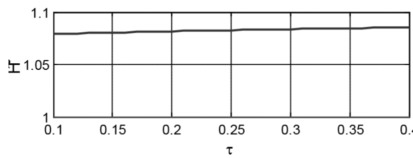


Figure 17.3: The relationship between growth and τ where $\delta_1 > \delta_2$ and $\alpha = 0.6$

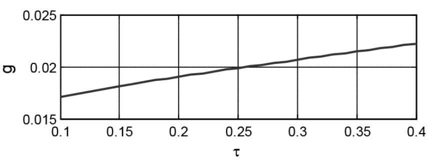


Figure 17.4: The relationship between growth and τ where $\delta_1 > \delta_2$ and $\alpha = -0.6$

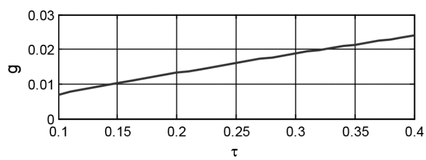


Figure 17.5: The relationship between inequality and τ where $\delta_1 > \delta_2$ and $\alpha = 0.6$

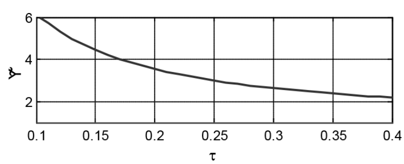
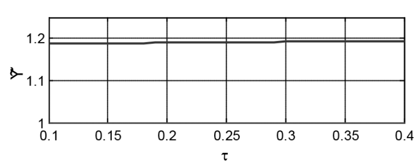


Figure 17.6: The relationship between inequality and τ where $\delta_1 > \delta_2$ and $\alpha = -0.6$



From the results depicted above, we can thus summarize our main findings as follow: the way heterogeneity is introduced in a model where private labor-time and public spending are inputs of individuals' human capital accumulation process matters regarding the implications of a change in the policy instrument on inequality. Moreover, the impacts of the size of public policy on growth and inequality depend on the degree of substitutability/complementarity between the two types of inputs in specific cases.[7] In particular,

[7] For the implication on growth, we note that the result of positive impact on growth does not imply a monotonic relationship between growth and policy instrument. Conversely, it implies that the direct effect of public policy on growth is likely to dominate the indirect one through private investment in human capital accumulation under heterogeneity (see section 3.2.3.1).

the results in the cases of heterogeneity in learning-abilities and motivation are consistent with Glomm and Kaganovich (2003): an additional size of public funding for education increases (reduces) inequality when private and public investments are complements (substitutes).

However, there is a great difference compared to the symmetry case, especially when private and public investments are substitutes: the policy instrument has a negative impact on growth in the symmetry case while it has a small positive impact in the case of heterogeneity. The reason is that the reduction of inequality gaps between individuals dissolves the negative impact on their labor-time allocated to human capital accumulation. According to quantitative results, this is still consistent with Blankenau and Simpson (2004): a positive growth effect of public educational expenditure is greater when private and public investments are complements than when they are substitutes.

3.3.4 Trade-off between equality and growth

In this section, we analyze the relations between income inequality, $\tilde{Y}$, and growth, g, when $0 < \alpha < 1$ and $\alpha < 0$. To proceed, we use the system of six equations (17)-(22) to numerically find the relations between $\tilde{Y}$ and g. Figures 18.1-18.6 illustrate the relations between inequality, $\tilde{Y}$, and growth, g, for three different kinds of heterogeneity: (i) skills heterogeneity ($A_1 > A_2$, $\phi_1 = \phi_2$ and $\delta_1 = \delta_2$) in Figures 18.1-18.2, (ii) learning-abilities heterogeneity ($A_1 = A_2$, $\phi_1 > \phi_2$ and $\delta_1 = \delta_2$) in Figures 18.3-18.4 and (iii) motivation heterogeneity ($A_1 = A_2$, $\phi_1 = \phi_2$ and $\delta_1 < \delta_2$) in Figures 18.5-18.6.

Figure 18.1: The relationship between inequality and growth where $A_1 > A_2$ and $\alpha = 0.6$

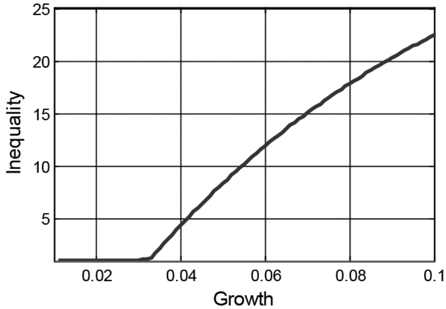


Figure 18.2: The relationship between inequality and growth where $A_1 > A_2$ and $\alpha = -0.6$

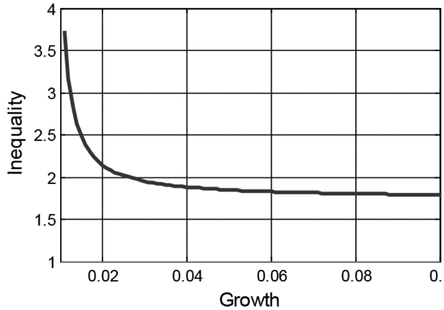


Figure 18.3: The relationship between inequality and growth where $\phi_1 > \phi_2$ and $\alpha = 0.6$

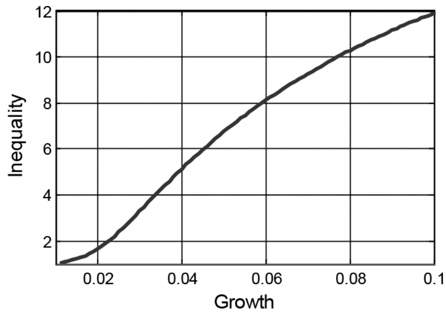


Figure 18.4: The relationship between inequality and growth where $\phi_1 > \phi_2$ and $\alpha = -0.6$

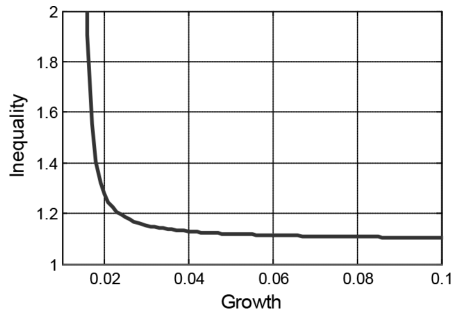


Figure 18.5: The relationship between inequality and growth where $\delta_1 > \delta_2$ and $\alpha = 0.6$

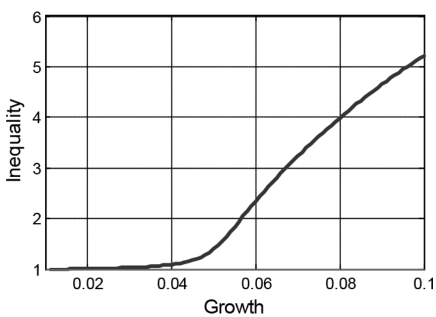
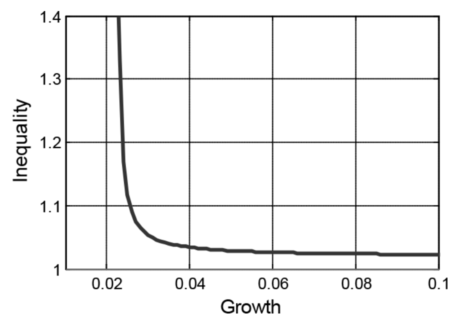


Figure 18.6: The relationship between inequality and growth where $\delta_1 > \delta_2$ and $\alpha = -0.6$



The results show that, for all kinds of heterogeneity, there is a trade-off between equality and economic growth when private and public investments in human capital accumulation are substitutes, while there is no such trade-off when they are complements. The reason is that, if the reduction in heterogeneity causes a fixed amount of percentage decrease in inequality gap for both substitutability and complementarity cases, it then causes a significant lower amount of percentage increase in growth in the case of substitutability than that of complementarity (see Table 4). This thus shows that the relations between inequality, $\tilde{Y}$, and growth, g, depend on the degree of substitutability/complementarity between the two types of educational investments, α.

3.3.5 Welfare implications in a democratic system

Given the fact that individuals are heterogeneous, they would not necessarily agree on the same level of tax rate, τ, (i.e., the size of the government). In this section, we point out a welfare conflict of interest: individuals of group 1 Vs. individuals of group 2. To determine the conflict, we assume the existence of a democratic system so that τ is chosen by the median voter who is referenced by $i = 1,2$.

Assuming that the economy is in steady state, the lifetime utility of individual i is given by:

$$U_i = \frac{\ln[(1-\tau)L_i^y A_i H_{i,0} + \delta_i(T - L_i^y - L_i^H)}{\rho} + \frac{g}{\rho^2}.$$

Taking the derivative of this expression with respect to τ yields:

$$\rho\frac{dU_i}{d\tau} = -\frac{1}{1-\tau} + \frac{1}{\rho}\frac{dg}{d\tau} - \delta_i\frac{dL_i^H}{d\tau}. \tag{23}$$

The first two terms of the right hand side of (23) shows effects that are common to every individual. The first term represents the consumption loss of a greater tax rate. The second term is a growth effect of such policy change. There is, in addition, an effect that corresponds to the effect of public expenditure, τ, on individual i, represented by the last term of the right hand side. It shows the impact of the policy instrument on labor-time of individual i devoted to human capital accumulation, L_i^H. Note that, under heterogeneity, the effects of τ on growth is always positive to guarantee that even the case of complementarity, a solution for $dU_i/d\tau = 0$ still exists.

To determine the choice of tax rate by the median voter, as before, we distinguish three cases: (i) skills heterogeneity ($A_1 > A_2$, $\phi_1 = \phi_2$ and $\delta_1 = \delta_2$), (ii) learning-abilities heterogeneity($A_1 = A_2$, $\phi_1 > \phi_2$ and $\delta_1 = \delta_2$) and (iii) motivation heterogeneity ($A_1 = A_2$, $\phi_1 = \phi_2$ and $\delta_1 < \delta_2$) For each case, expression (23) allows us to compute what the choice of tax rate is and how the levels of tax rates chosen differ if the median voter belongs to group 1 and 2. To proceed, we use the benchmark parameter values from Table 2 to numerically find the results.

Figure 19.1 shows that individuals of group 1 prefer the same tax rate as those of group 2 in the case of skills heterogeneity. The explanation is that, as the tax rate increases, the effects on labor-time of the two types of individuals are equivalent whether private and public investments are either substitutes or complements. However, Figures 19.2-19.3 illustrate that a level of tax chosen by individuals of group 1 is higher than that chosen by those of group 2 in the cases of heterogeneity in learning abilities and motivation. This is because the reduction (increase) in labor supply of individuals of group1 is greater (smaller) than the increase in that of group 2 when the two types of investments are substitutes (complements), see equation (23) and Figures 14.1-14.6 and 16.1-16.6. Moreover, the result shows that the gap of the chosen tax choice by individuals of group 1 and 2 is wider when the two types of investments become more substitutable. This is because an increase in tax rate causes a larger gap in inequality between individuals when investments are more substitutable and individuals are differentiated in learning abilities and motivation as mentioned in sections 3.3.3.2 and 3.3.3.3. As a result, how the levels of tax rates chosen by the two different types of individuals does not depend on the degree of substitutability/complementarity between private and public investments.

To analyze the welfare analysis related to the implications on growth and inequality, we use the results from Figures 19.1-19.3 together with those depicted in section 3.3.3. A brief summary of results is given in Table 6 and discussed in the next subsections.

Figure 19.1: The choice of tax rate chosen by individuals of group i in the case of skills heterogeneity with respect to α

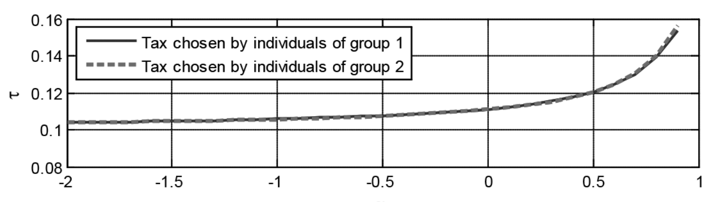


Figure 19.2: The choice of tax rate chosen by individuals of group i in the case of learning-abilities heterogeneity with respect to α

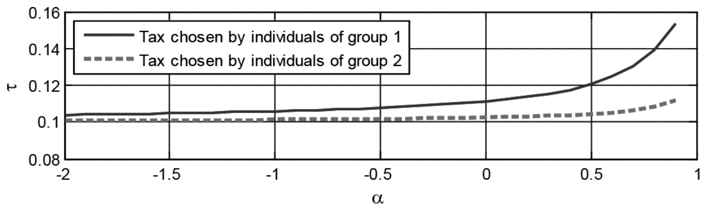


Figure 19.3: The choice of tax rate chosen by individuals of group i in the case of motivation heterogeneity with respect to α

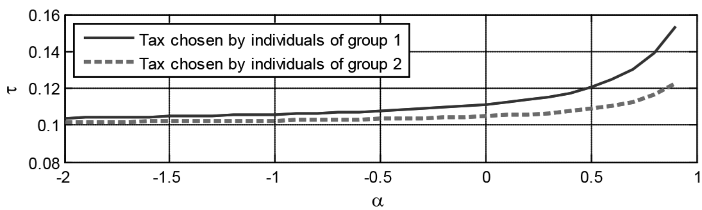


Table 6: Analysis of growth, inequality and welfare regarding to the choice of tax rate

Case	**Situation**	***dg/dr***	$d\tilde{Y}/d\tau$	$\tau_1 \lesseqgtr \tau_2$
Skills heterogeneity	$0 < \alpha < 1$	> 0	$= 0$	$=$
	$\alpha < 0$	> 0	$= 0$	$=$
Learning-abilities heterogeneity	$0 < \alpha < 1$	> 0	< 0	$>$
	$\alpha < 0$	> 0	> 0	$>$
Motivation heterogeneity	$0 < \alpha < 1$	> 0	< 0	$>$
	$\alpha < 0$	> 0	> 0	$>$

3.3.5.1 The case of skills heterogeneity in the democratic system

As mentioned previously, the effect on growth is common to every individual, $dg/d\tau > 0$ thus has no impact on the choice of tax rate (see equation (23) and Table 6). Moreover, a greater level of tax rate does not affect inequality because the effects on the quantity of labor-time devoted to human capital accumulation for both types of individuals are equivalent as the tax rate increases (see section 3.3.3.1). Thus, we have $\tau_1 = \tau_2$ and $d\tilde{Y}/d\tau = 0$, respectively. As a result, a higher-skilled individual prefers the same levels of tax, growth and inequality as those of lower skill when private and public investments in education are either substitutes or complements.

3.3.5.2 The case of learning-abilities heterogeneity in the democratic system

As before, $dg/d\tau > 0$ does not affect the choice of tax rate. However, in this case, the negative (positive) effect of tax on labor-time devoted to human capital accumulation is more pronounced for higher-ability (lower-ability) individuals when private and public investments are substitutes (complements) (see section 3.3.3.2). This leads to $\tau_1 > \tau_2$ (see equation (23) and Table 6). In addition to the previous effect, an increase in tax rate causes a narrowing (expanding) gap in labor-time between the two groups of individuals when investments are substitutes (complements). Thus, we have $d\tilde{Y}/d\tau <$ (>)0 when $0 < \alpha < 1$ ($\alpha < 0$). As a result, a higher-ability individual prefers higher tax and growth rates compared to an individual of lower ability regardless of the degree of substitutability/ complementarity. However, the former prefers a lower (higher) level of inequality than the latter when the two types of investments are substitutes (complements).

3.3.5.3 The case of motivation heterogeneity in the democratic system

From Table 6, all the results are the same as in section 3.3.5.2. Thus, we conclude that a more-motivated individual prefers higher levels of tax rate and growth rate than a less-motivated individual when private and public investments in education are either substitutes or complements. However, the former prefers a lower (higher) level of inequality than the latter when private and public investments are substitutes (complements).

The intuitions are similar to the previous case, except when private and public investments in human capital accumulation are complements. As mentioned in section 3.3.3.3, a rise in tax rate causes a higher (lower) quantity of labor supply of the more-motivated (higher-ability) individuals than those of less motivated (lower ability). However, the effect of tax through motivation heterogeneity dominates that through the narrowing gap in labor supply. In this case, we thus still have $d\tilde{Y}/d\tau <$ (>)0 when $0 < \alpha < 1$ ($\alpha < 0$).

4. Conclusion

We have developed an endogenous growth model with human capital accumulation which is the outcome of both private and public investments via a CES production function. This allowed us to emphasize the effects of the degree of substitutability/complementarity between private and public investments on growth, inequality and welfare.

Under symmetry, we have found that there is a positive relationship between growth and public expenditure on human capital when private and public investments are complements. However, when the two types of investments are substitutes, there can be a negative relationship. Interestingly, we have shown that not only the degree of substitutability/complementarity between the two types of educational investments plays an important role in determining the policy effects on inequality, but also, it has been revealed that the level of the welfare-maximizing tax is positively related with the degree of substitutability/complementarity between the two types of investments. Under heterogeneity, the noteworthy result was that, in addition to the degree of substitutability/complementarity, the policy effect on inequality and individuals' welfare crucially depends on the way individuals are differentiated.

For future research, some extensions are possible. In particular, it could be interesting to analyze how a change in the nature of private investment in human capital accumulation affects the results we have derived here. For instant, we could assume that individuals use material resources (i.e., output as private investment in human capital). This could raise an interesting issue regarding the significance of the substitutability/complementarity between private and public educational investments. This is on our agenda for future work.

5. Appendix

5.1 No transitional dynamics in the case of symmetry

Under symmetry, we have $\overline{Y}_t = Y_t$, $A_1 = A_2$, $\phi_1 = \phi_2$ and $\delta_1 = \delta_2$. Using the first-order condition (6), (7) and the technology of human capital (2), we obtain:

$$\mu_t = \frac{\delta(L_t^H)^{1-\alpha}[\beta(L_t^H)^\alpha + (1-\beta)(\tau A L^y)^\alpha]^{1-1/\alpha}}{H_t \beta \varphi} \tag{24}$$

and

$$\rho = \left(\frac{1+\delta L^H}{\delta L^H}\right) \frac{\beta(L_t^H)^\alpha}{[\beta(L_t^H)^\alpha + (1-\beta)(\tau A L^y)^\alpha]} \frac{\dot{H}_t}{H_t} + \frac{\dot{\mu}_t}{\mu_t}. \tag{25}$$

Using (24) and (25) yields:

$$\frac{\dot{\mu}_t}{\mu_t} = \left\{1 - \frac{\beta(L_t^H)^\alpha}{[\beta(L_t^H)^\alpha + (1-\beta)(\tau A L^y)^\alpha]}\right\} \frac{\dot{L}_t^H}{L_t^H} - \varphi[\beta(L_t^H)^\alpha + (1-\beta)(\tau A L^y)^\alpha]^{1/\alpha}. \tag{26}$$

Substituting expression (26) into (25), we derive the growth rate of labor-time devoted to human capital accumulation, which is a function of L_t^H:

$$\frac{\dot{L}_t^H}{L_t^H} = \frac{\rho - \varphi[\beta(L_t^H)^\alpha + (1-\beta)(\tau A L^y)^\alpha]^{1/\alpha}\left[\left(\frac{1+\delta L^H}{\delta L^H}\right)\frac{\beta(L_t^H)^\alpha}{\beta(L_t^H)^\alpha + (1-\beta)(\tau A L^y)^\alpha} - 1\right]}{(1-\alpha)\left[\frac{(1-\beta)(\tau A L^y)^\alpha}{\beta(L_t^H)^\alpha + (1-\beta)(\tau A L^y)^\alpha}\right]}.$$

For simplicity, we use the above expression to numerically show that there is no transitional dynamics. Figures 20.1-20.2 illustrate that the differential equation giving $\dot{L}_t^H / L_t^H$ as a function of L_t^H describes an unstable process. That is, as $\dot{L}_t^H = 0$ at each instant, the economy jumps immediately to the steady state.

Figure 20.1: Transitional Dynamics under Symmetry where α = 0.6

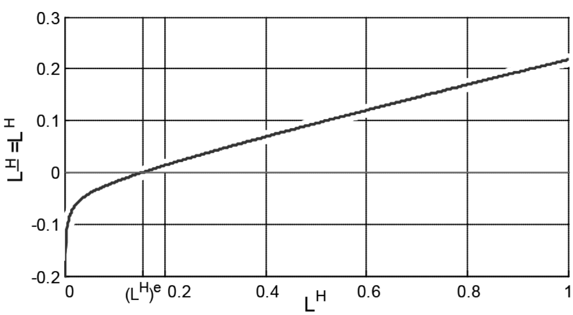
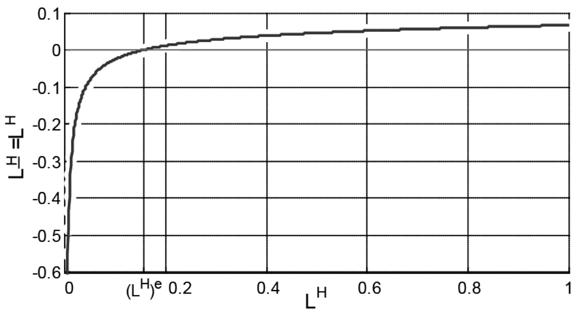


Figure 20.2: Transitional Dynamics under Symmetry where α = –0.6



5.2 Proof of proposition

At steady state under symmetry, we have $H_t = H$, $Y_t = \overline{Y}_t = Y$, $L^y_{i,t} = L^y$, $L^H_{i,t} = L_H$. From the first-order conditions, we derive . As a result, equation (5) obtains (8). Then we rearrange the first-order equations (4)-(7) and the production of human capital (2) to obtain an implicit function:

$$f = \Delta[(L^H)^e] - \Psi[(L^H)^e] = 0,$$

where

$$\Delta[(L^H)^e] = \frac{\phi}{\delta}\{\beta[\delta(L^H)^e]^\alpha + (1-\beta)(\tau A)^\alpha\}^{\frac{1}{\alpha}},$$

and

$$\Psi[(L^H)^e] = \rho\left\{\frac{\beta[\delta(L^H)^e]^{1+\alpha} + (1-\beta)(\tau A)^\alpha\delta(L^H)^e}{\beta[\delta(L^H)^e]^\alpha - (1-\beta)(\tau A)^\alpha\delta(L^H)^e}\right\}.$$

The implicit function is to solve for a unique growth rate of consumption, output, and human capital, g^e, and a unique amount of equilibrium education time, $(L^H)^e$. To proceed, we take the first derivative of Δ(.) with respect to $(L^H)^e$ to obtain:

$$\frac{d\Delta(.)}{d(L^H)^e} = \frac{\Delta(.)}{(L^H)^e}\left(\frac{z}{\Omega}\right) > 0, \tag{27}$$

where $z = \beta[\delta(L^H)^e]^\alpha$, $q = (1-\beta)(\tau A)^\alpha$, and $\Omega = z + q$. Then, the second derivative of Δ(.) with respect to $(L^H)^e$ is given by:

$$\frac{d^2\Delta(.)}{d[(L^H)^e]^2} = \frac{(1-\alpha)}{(L^H)^e}\frac{d\Delta(.)}{d(L^H)^e}\left(\frac{z}{\Omega} - 1\right) < 0.$$

Since $z/\Omega < 1$, the second derivative of $\Delta(.)$ is negative. Thus, $\Delta(.)$ depicts strictly increasing and strictly concave function of labor-time devoted to human capital accumulation, $(L^H)^e$. However, for $\Psi(.)$, we have:

$$\frac{d\Psi(.)}{d(L^H)^e} = \frac{\Delta(.)}{(L^H)^e}\left[\frac{(1-\alpha)z+q}{\Omega} - \frac{\alpha z - q\delta(L^H)^e}{z - q\delta(L^H)^e}\right] > 0, \tag{28}$$

and

$$\frac{d^2\Psi(.)}{d[(L^H)^e]^2} = \{\alpha(\alpha-1)(\beta+q)[\delta(L^H)^e]^\alpha + (2-3\alpha+\alpha^2)\Omega\delta(L^H)^e\}zq.$$

For expression (28), the first term on the right hand side is greater than one while the second term is less than one. This shows that $\Psi(.)$ also has an upward slope. Notice that the sign of the second derivative of $\Psi(.)$ depends on the value of α. If $\alpha < 0$, we derive a positive sign and $\Psi(.)$ depicts strictly increasing and strictly convex function (see Figure 1.2). But if $0 < \alpha < 1$, the second derivative of $\Psi(.)$ is negative so that $\Psi(.)$ depicts strictly increasing and strictly concave function $(L^H)^e$ (see Figure 1.1).

For the case of $\alpha < 0$, there obviously exists an intersection between the slopes of $\Delta(.)$ and $\Psi(.)$ deriving a unique amount of labor time allocated to human capital accumulation, $(L^H)^e$, and a unique growth rate, g^e, in the range of $g \in (0,1)$ and $L^H \in (0,1)$.

For the case of $0 < \alpha < 1$, the assumptions $\delta > 0$ and $L^H > 0$ give the result that the slope of $\Delta(.)$ is flatter than those of $\Psi(.)$. Thus, the slopes intercept only once in the range of $g \in (0,1)$ and $L^H \in (0,1)$, guaranteeing the uniqueness of the steady-state solution.

5.3 Taylor series approximation

In this section, we find the relationship between the unknowns; $(L^H)^e$ and g^e, and the parameters of the model. To proceed, we rearrange expressions (6) and (7) to obtain:

$$\frac{g^e+\rho}{g^e} = \left[\frac{1+\delta(L^H)^e}{\delta(L^H)^e}\right]\left\{\frac{\beta[\delta(L^H)^e]^\alpha}{\beta[\delta(L^H)^e]^\alpha + (1-\beta)(\tau A)^\alpha}\right\}, \tag{29}$$

and rearrange equations (5) and (12) and obtain:

$$g^e = \frac{\phi}{\delta}\{\beta[\delta(L^H)^e]^\alpha + (1-\beta)(\tau A)^\alpha\}^{1/\alpha}. \tag{30}$$

Since equation (5) shows that the steady-state equilibrium amount of labor-time devoted to production of output, $(L^y)^e$, is determined by the marginal disutility of non-leisure time, δ, we will leave $(L^y)^e$ and put it aside. Up to this point, we have two equations, (29) and (30); and two unknowns, $(L^H)^e$ and g^e. Then, we use the first-order Taylor series approximation to equations (29) and (30) around the steady state in terms of elasticities as follows:

$$\begin{aligned} df_1 = &-\frac{z - q\delta(L^H)^e}{[1+\delta(L^H)^e]z}\left(\frac{dg^e}{g^e}\right) + \left\langle\frac{z + q\{1-\alpha[1+\delta(L^H)^e]\}}{[1+\delta(L^H)^e]\Omega}\right\rangle\left(\frac{d(L^H)^e}{(L^H)^e}\right) \\ &+ \left\langle\frac{z + q\{1-\alpha[1+\delta(L^H)^e]\}}{[1+\delta(L^H)^e]\Omega}\right\rangle\left(\frac{d\delta}{\delta}\right) + \frac{z - q\delta(L^H)^e}{[1+\delta(L^H)^e]z}\left(\frac{d\rho}{\rho}\right) \\ &+ \frac{\alpha q}{\Omega}\left(\frac{dA}{A}\right) + \frac{\alpha q}{\Omega}\left(\frac{d\tau}{\tau}\right) = 0, \end{aligned}$$

$$\begin{aligned} df_2 = &\left(\frac{dg^e}{g^e}\right) - \frac{z}{\Omega}\left(\frac{d(L^H)^e}{(L^H)^e}\right) + \frac{q}{\Omega}\left(\frac{d\delta}{\delta}\right) - \left(\frac{d\phi}{\phi}\right) \\ &- \frac{q}{\Omega}\left(\frac{dA}{A}\right) - \frac{q}{\Omega}\left(\frac{d\tau}{\tau}\right) = 0. \end{aligned}$$

We then derive the standard matrix form: $Dx = b_k$, where $k = 1,2,\ldots,5$ represents the matrix of each variable multiplied by its own elasticity,

$$D = \begin{bmatrix} -\frac{z - q\delta(L^H)^e}{[1+\delta(L^H)^e]z} & \left\langle\frac{z + q\{1-\alpha[1+\delta(L^H)^e]\}}{[1+\delta(L^H)^e]\Omega}\right\rangle \\ 1 & -\frac{z}{\Omega} \end{bmatrix},\ x = \begin{bmatrix} \frac{dg^e}{g^e} \\ \frac{d(L^H)^e}{(L^H)^e} \end{bmatrix},$$

$$b_1 = \begin{bmatrix} -\frac{z + q\{1-\alpha[1+\delta(L^H)^e]\}}{[1+\delta(L^H)^e]\Omega} \\ -\frac{q}{\Omega} \end{bmatrix}\left[\frac{d\delta}{\delta}\right],\ b_2 = \begin{bmatrix} -\frac{z - q\delta(L^H)^e}{[1+\delta(L^H)^e]z} \\ 0 \end{bmatrix}\left[\frac{d\rho}{\rho}\right]$$

$$b_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\left[\frac{d\varphi}{\varphi}\right],\ b_4 = \begin{bmatrix} -\frac{\alpha q}{\Omega} \\ \frac{q}{\Omega} \end{bmatrix}\left[\frac{dA}{A}\right],\ b_5 = \begin{bmatrix} -\frac{\alpha q}{\Omega} \\ \frac{q}{\Omega} \end{bmatrix}\left[\frac{d\tau}{\tau}\right].$$

Using Cramer's rule, we obtain the relationships of each pair between the unknowns and parameters. Since we assume $\alpha < 1$, the determinant of the D-matrix is not equal to zero: $|D| = (1-\alpha)(L^y)^e q/\Omega \neq 0$. This prevents all undefined solutions and meets the requirement of a nonsingular matrix. Under matrix D, we can find some relationships between the unknowns and parameters. The relationships between the steady-state growth rate and parameters are:

$$\frac{dg^e}{d\delta}\frac{\delta}{g^e} = -\frac{z + q\{1-\alpha[1+\delta(L^H)^e]\}}{(1-\alpha)[1+\delta(L^H)^e]} < 0.$$

Since there exists a positive unique amount of equilibrium growth rate, $g^e > 0$, we must have $\beta(L_t^H)^\alpha + (1-\beta)(\tau A L^y)^\alpha > 0$ (or $\Omega > 0$), see equation (12). Thus, the numerator on the right hand side of the above relationship, $z + q\{1-\alpha[1+\delta(L^H)^e]\}$, is positive. Therefore, the above expression shows a negative relationship between g^e and δ.

$$\frac{dg^e}{d\rho}\frac{\rho}{g^e} = -\frac{z - q\delta(L^H)^e}{(1-\alpha)[1+\delta(L^H)^e]q} < 0.$$

From equation (11), we also need $z - q\delta(L^H)^e > 0$ to have a positive equilibrium growth rate. Thus, g^e is negatively related to ρ but is positively related to ϕ:

$$\frac{dg^e}{d\phi}\frac{\phi}{g^e} = \frac{z + q\{1-\alpha[1+\delta(L^H)^e]\}}{(1-\alpha)[1+\delta(L^H)^e]q} > 0.$$

However, the relationships between the growth rate and the remaining parameters are ambiguous. We use numerical methods to find the signs of the relationships using the following expressions:

$$\frac{dg^e}{dA}\frac{A}{g^e} = \frac{\{1-\alpha[1+\delta(L^H)^e]\}}{(1-\alpha)[1+\delta(L^H)^e]}, \tag{31}$$

$$\frac{dg^e}{d\tau}\frac{\tau}{g^e} = \frac{\{1-\alpha[1+\delta(L^H)^e]\}}{(1-\alpha)[1+\delta(L^H)^e]}. \tag{32}$$

Notice that the solutions from (31) and (32) depend on the value of α. If $\alpha < 0$, the signs of dg^e/dA and $dg^e/d\tau$ will be positive. However, the analysis is complex when $0 < \alpha < 1$ as discussed in section 3.2.2 and 3.2.3.

The relationships between $(L^H)^e$ and parameters are:

$$\frac{d(L^H)^e}{\delta}\frac{\delta}{(L^H)^e} = -\frac{[z - q\delta(L^H)^e]q/z + z + q\{1 - \alpha[1 + \delta(L^H)^e]\}}{(1-\alpha)[1 + \delta(L^H)^e]q} < 0,$$

$$\frac{d(L^H)^e}{\rho}\frac{\rho}{(L^H)^e} = -\frac{[z - q\delta(L^H)^e]\Omega}{(1-\alpha)[1 + \delta(L^H)^e]zq} < 0,$$

$$\frac{d(L^H)^e}{\phi}\frac{\phi}{(L^H)^e} = \frac{[z - q\delta(L^H)^e]\Omega}{(1-\alpha)[1 + \delta(L^H)^e]zq} > 0.$$

Again, the unsigned relationships will be numerically calibrated by using the following expressions:

$$\frac{d(L^H)^e}{dA}\frac{A}{(L^H)^e} = \frac{z\{1 - \alpha[1 + \delta(L^H)^e]\} - q\delta(L^H)^e}{(1-\alpha)[1 + \delta(L^H)^e]z}, \tag{33}$$

$$\frac{d(L^H)^e}{d\tau}\frac{\tau}{(L^H)^e} = \frac{z\{1 - \alpha[1 + \delta(L^H)^e]\} - q\delta(L^H)^e}{(1-\alpha)[1 + \delta(L^H)^e]z}. \tag{34}$$

Rearranging expression (33) and (34), we obtain:

$$\frac{d(L^H)^e}{dA}\frac{A}{(L^H)^e} = \left(\frac{1}{1-\alpha}\right)\left(\frac{\rho}{g^e + \rho} - \alpha\right),$$

$$\frac{d(L^H)^e}{d\tau}\frac{\tau}{(L^H)^e} = \left(\frac{1}{1-\alpha}\right)\left(\frac{\rho}{g^e + \rho} - \alpha\right).$$

These show that the signs of $d(L^H)^e/dA$ and $d(L^H)^e/d\tau$ depend on the value of α. If $\alpha < 0$, we obtain positive signs. Again, the analysis is complex when $0 < \alpha < 1$. We let $\hat{\alpha}$ be the value of the degree of substitutability/complementarity so that $d(L^H)^e/dA = d(L^H)^e/d\tau = 0$. If $0 < \alpha < \hat{\alpha}$, we have $d(L^H)^e/dA > 0$ and $d(L^H)^e/d\tau > 0$. If $0 < \hat{\alpha} < \alpha < 1$, we have $d(L^H)^e/dA < 0$ and $d(L^H)^e/d\tau < 0$.

5.4 Normalization of CES production of human capital

Cobb-Douglas function is a special form of the CES function in which the elasticity of substitution is constant at one. It is extensively used in the growth literature. Although, it satisfies the property of *dimensional homogeneity* due to the constant returns to scale assumption, it gives an artificial symmetry among its inputs. In other words, it lacks of sensitivity analysis on the values of key elasticities. Instead, the generalized CES function allows us to analyze variation of a degree of substitutability/complementarity between inputs. However, it raises the problem of dimensionality. Klump and de La Grandville (2000) introduced the concept of *normalization* (or re-parameterization) of the CES function in an analytical way to solve this problem which occurs when input factors are measured in different units. Since efficiency and distributional parameters (i.e., A and β) are dimensional constant, their values will change by a different choice of units of degree of substitutability/complementarity, α. This helps satisfy the normalized CES function which is dimensionless.

Following Klump and de La Grandville (2000), we then normalize CES production function of human capital in the steady state to calibrate our model. First, we define a steady-state baseline point (normalization point) as shown in Table 2. Then, we use the baseline point to find new efficiency values of parameter ϕ and distributional values of parameter β for each different choice of units of α. Using the production of human capital, we obtain the growth rate of human capital:

$$\gamma_H = \frac{\phi\tau A}{\delta}\left[\beta\left(\frac{L^H}{\tau AL^y}\right)^{\alpha} + (1-\beta)\right]^{1/\alpha}.$$

Let $\Gamma = L^H/\tau AL^y$ be investments ratio, $\Phi = \phi\tau A/\delta$ and $\eta = (d\gamma_H/d\Gamma)(\Gamma/\gamma_H)$. We rearrange the above equation to obtain:

$$\gamma_H = \Phi[\beta\Gamma^{\alpha} + (1-\beta)]^{1/\alpha}, \tag{35}$$

$$\eta = \frac{\beta\Gamma^{\alpha}}{\beta\Gamma^{\alpha} + (1-\beta)}.$$

After manipulations, we have:

$$\beta = \frac{\eta\Gamma^{-\alpha}}{\eta\Gamma^{-\alpha} + (1-\eta)}, \tag{36}$$

$$\Phi = \gamma_H[\eta\Gamma^{-\alpha} + (1-\eta)]^{1/\alpha}. \tag{37}$$

Using equation (35)-(37), we obtain a normalized growth rate of human capital as a weighted mean of order α taken over normalized inputs, Γ/Γ_o:

$$\frac{\gamma_H}{\gamma_{Ho}} = \left[\eta_o\left(\frac{\Gamma}{\Gamma_o}\right)^{\alpha} + (1-\eta_o)\right], \tag{38}$$

where o denotes the baseline point. Note that the ratios γ_H/γ_{Ho} and Γ/Γ_o are dimensionless. After deriving the relevant equations, we then calibrate the model in the case where α varies, using the following steps: (i) we choose any value of α_j and Γ_o to find η_o, $\forall_j$, (ii) For given Γ_o and η_o, we obtain the new values of β_j and Φ_j for each different units of α_j, $\forall_j$, and (iii) we use β_j and Φ_j to calibrate our model where α varies. Since $\Gamma = \Gamma_o$ means that γ_H is independent of α, we assume that $\Gamma_j \neq \Gamma_o$ to avoid $\alpha = -\alpha$. As a result, we will obtain the implications on economic variables including Γ_j, $\forall_j$.

6. Reference

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